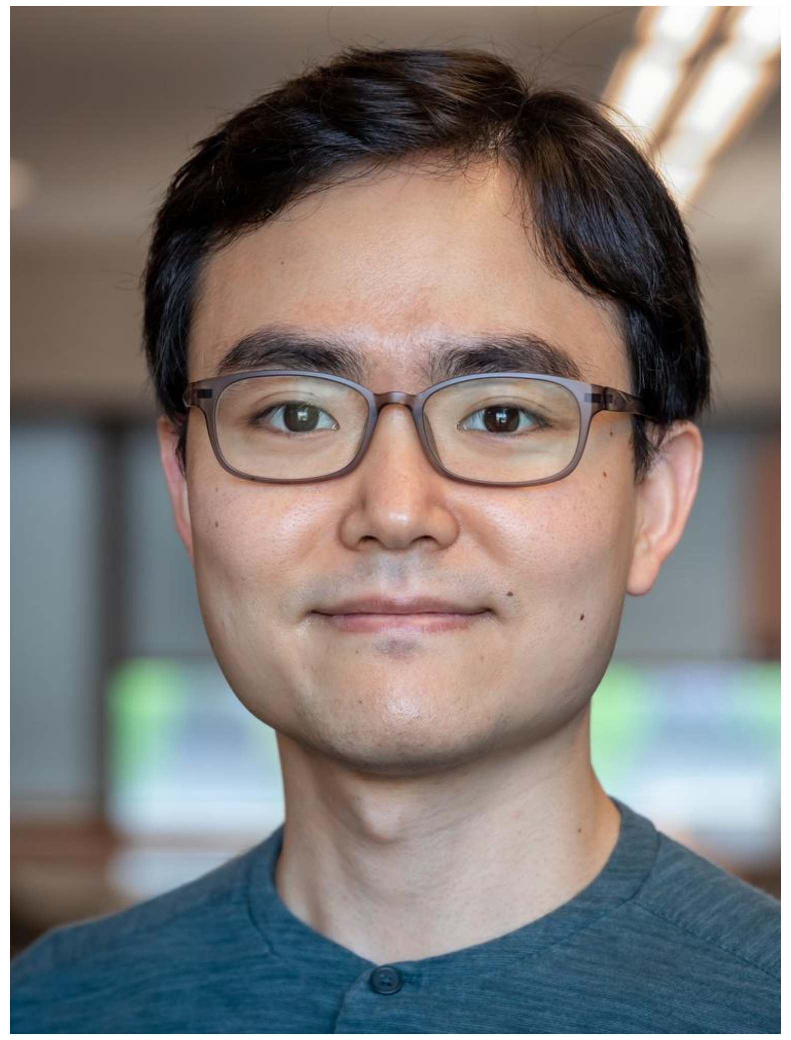
Bootstrapping the Ising Model on the Lattice

Victor A. Rodriguez
Princeton University

Positivity @ PCTS

based on 2206.12538 [hep-th] w/ Minjae Cho, Barak Gabai, Ying-Hsuan Lin, Joshua Sandor, Xi Yin



Minjae Cho

Barak Gabai

Ying-Hsuan Lin

Joshua Sandor

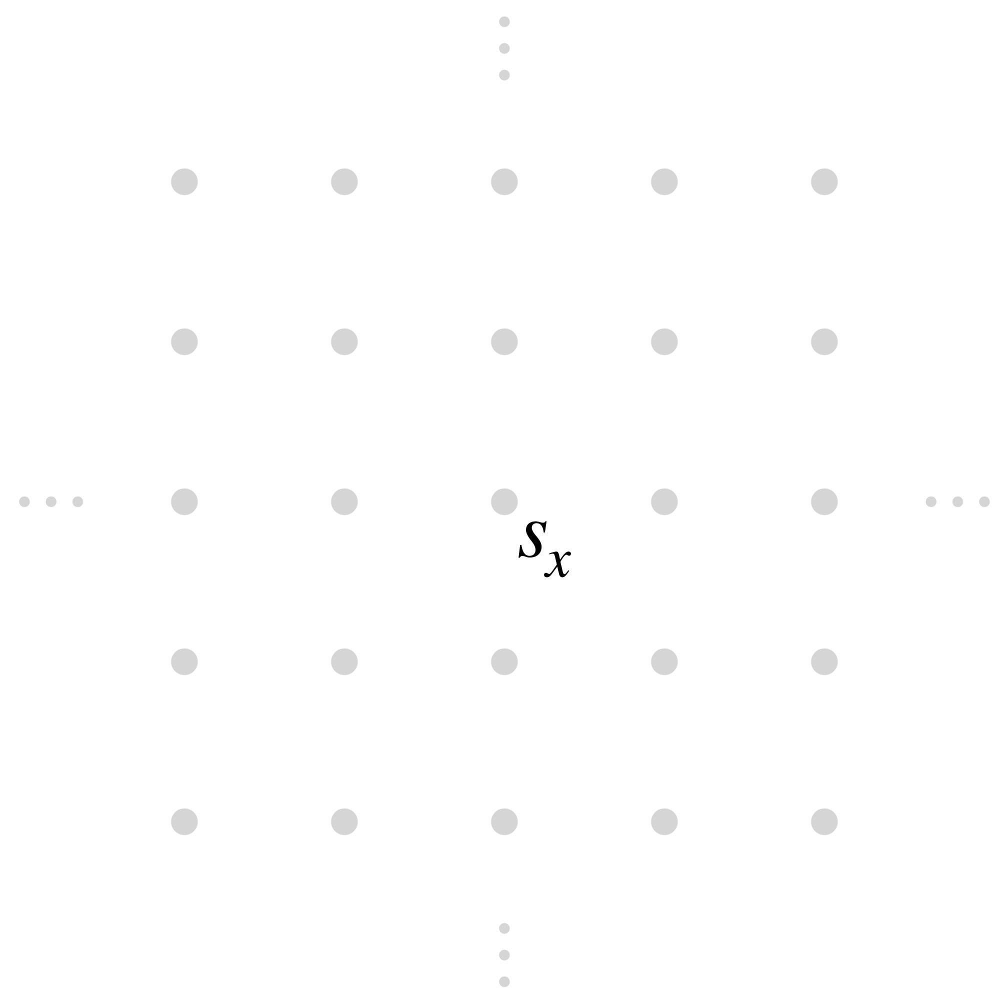
Xi Yin

Outline

- Ising model review
- Bootstrap of the Ising model
    1. Relation: “spin-flip” equations
    2. Positivity: reflection positivity, Griffiths inequalities, etc.
- Results in 2D and 3D Ising model
- Prospects

Ising Model

Physical system for intuition: magnets



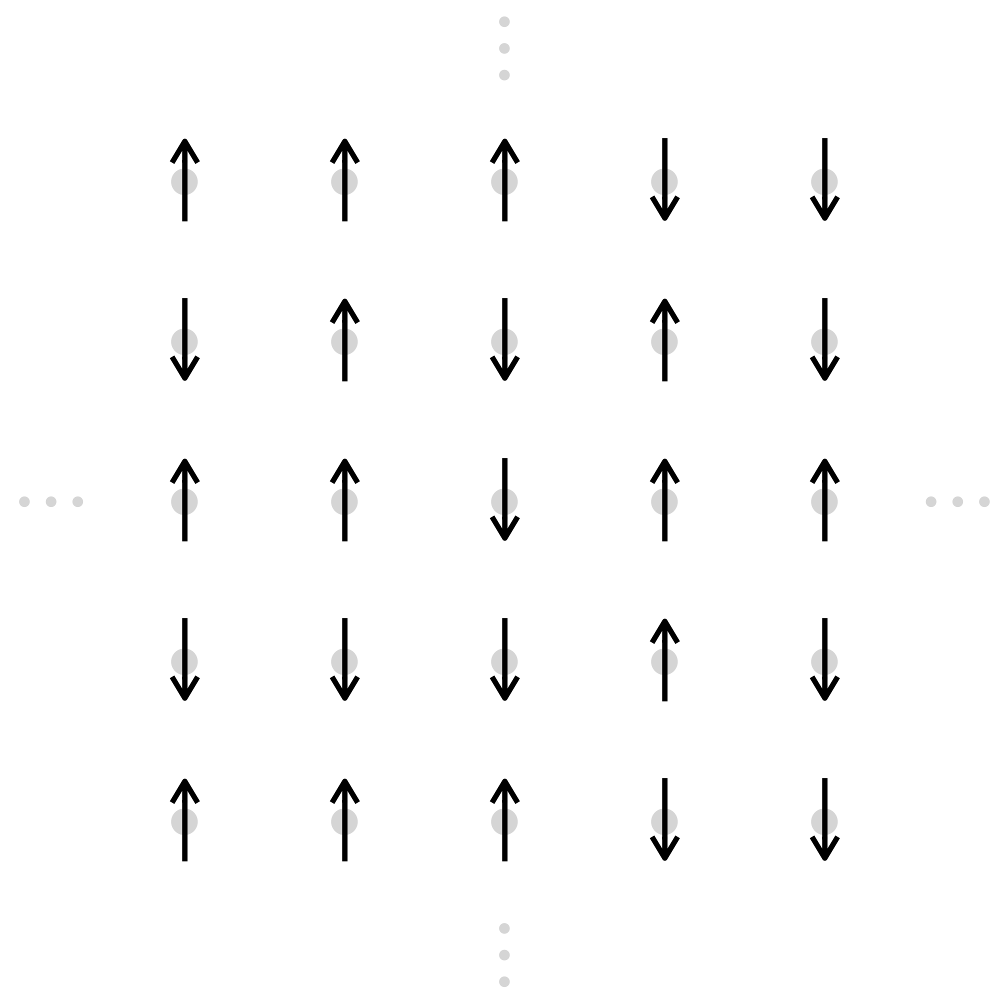
At each lattice site $x \in \Lambda$, the variable s_x (called "spin") can take either of two values

$$s_x = \begin{cases} 1 & \text{"spin up"} \quad \uparrow \\ -1 & \text{"spin down"} \quad \downarrow \end{cases}$$

electrons, tiny magnets

Ising Model

Physical system for intuition: magnets



At each lattice site $x \in \Lambda$, the variable s_x (called “spin”) can take either of two values

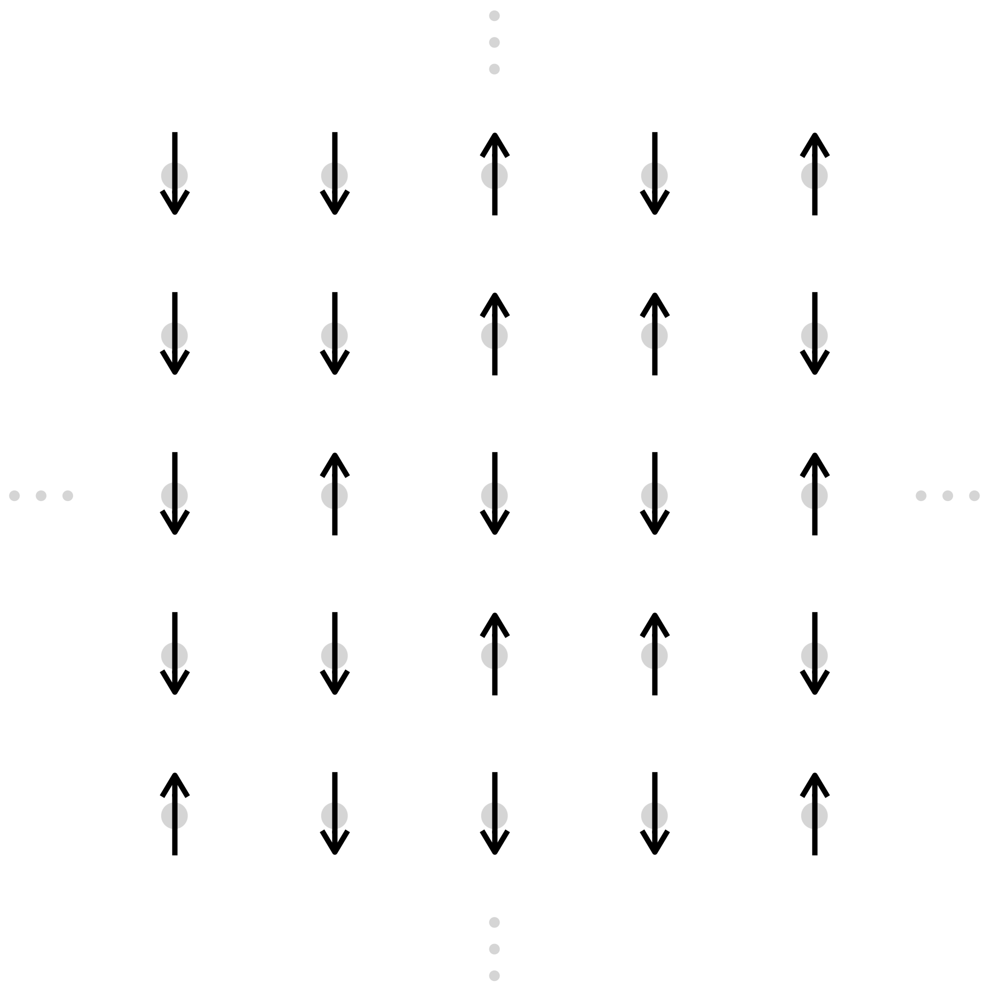
$$s_x = \begin{cases} 1 & \text{"spin up"} \quad \uparrow \\ -1 & \text{"spin down"} \quad \downarrow \end{cases}$$

electrons, tiny magnets

A spin configuration of the lattice system is a particular assignment of a spin value for each site.

Ising Model

Physical system for intuition: magnets



At each lattice site $x \in \Lambda$, the variable s_x (called "spin") can take either of two values

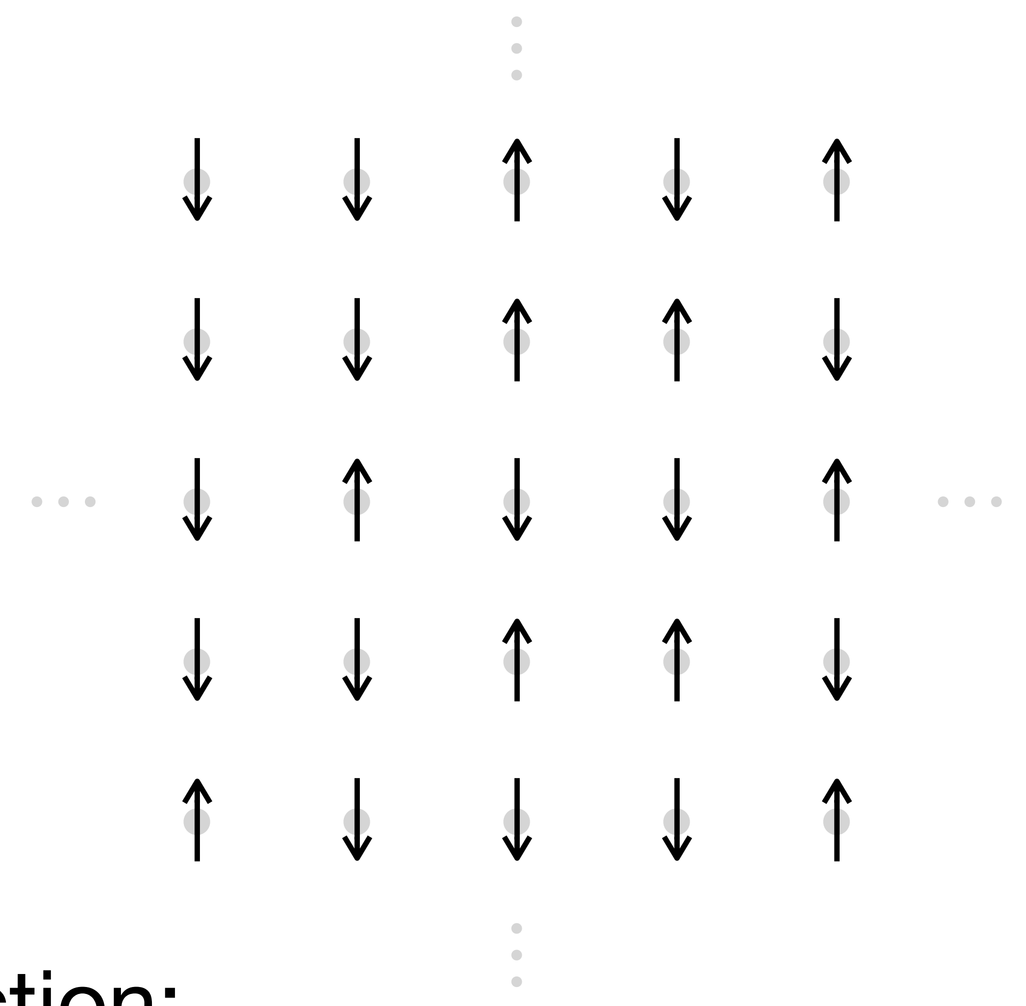
$$s_x = \begin{cases} 1 & \text{"spin up"} \quad \uparrow \\ -1 & \text{"spin down"} \quad \downarrow \end{cases}$$

electrons, tiny magnets

A spin configuration of the lattice system is a particular assignment of a spin value for each site.

Ising Model

Physical system for intuition: magnets



At each lattice site $x \in \Lambda$, the variable s_x (called "spin") can take either of two values

$$s_x = \begin{cases} 1 & \text{"spin up"} \quad \uparrow \\ -1 & \text{"spin down"} \quad \downarrow \end{cases}$$

electrons, tiny magnets

A spin configuration of the lattice system is a particular assignment of a spin value for each site.

Partition function:

$$Z = \sum_{\substack{\text{spin} \\ \text{configs}}} \exp\left[-\frac{1}{T} E\begin{pmatrix}\text{spin} \\ \text{config}\end{pmatrix}\right]$$

where the exponential is interpreted as the probability that the system is in that specific spin config

Ising Model

Physical system for intuition: magnets

For the Ising model,

$$E(\{s_x\}) = -J \sum_{\langle xy \rangle} s_x s_y - h \sum_{x \in \Lambda} s_x$$

where $\langle xy \rangle$ means $x, y \in \Lambda$ such that they are directly adjacent sites

Ising Model

Physical system for intuition: magnets

For the Ising model,

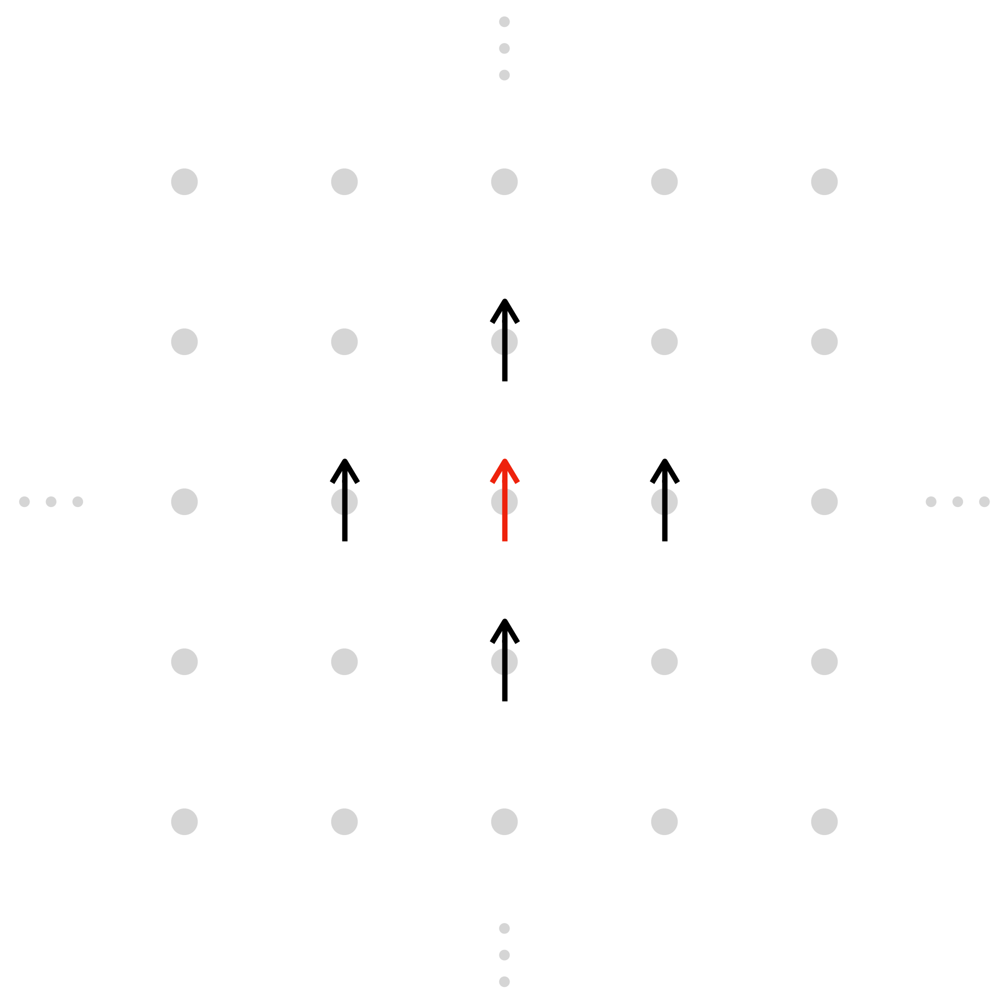
$$E(\{s_x\}) = -J \sum_{\langle xy \rangle} s_x s_y - h \sum_{x \in \Lambda} s_x$$

where $\langle xy \rangle$ means $x, y \in \Lambda$ such that they are directly adjacent sites

- 2nd term — external magnetic field h:
  spins want to point in the same direction as the external magnetic field
  (energetically favorable to do so)

Ising Model

Physical system for intuition: magnets



For the Ising model,

$$E(\{s_x\}) = -J \sum_{\langle xy \rangle} s_x s_y - h \sum_{x \in \Lambda} s_x$$

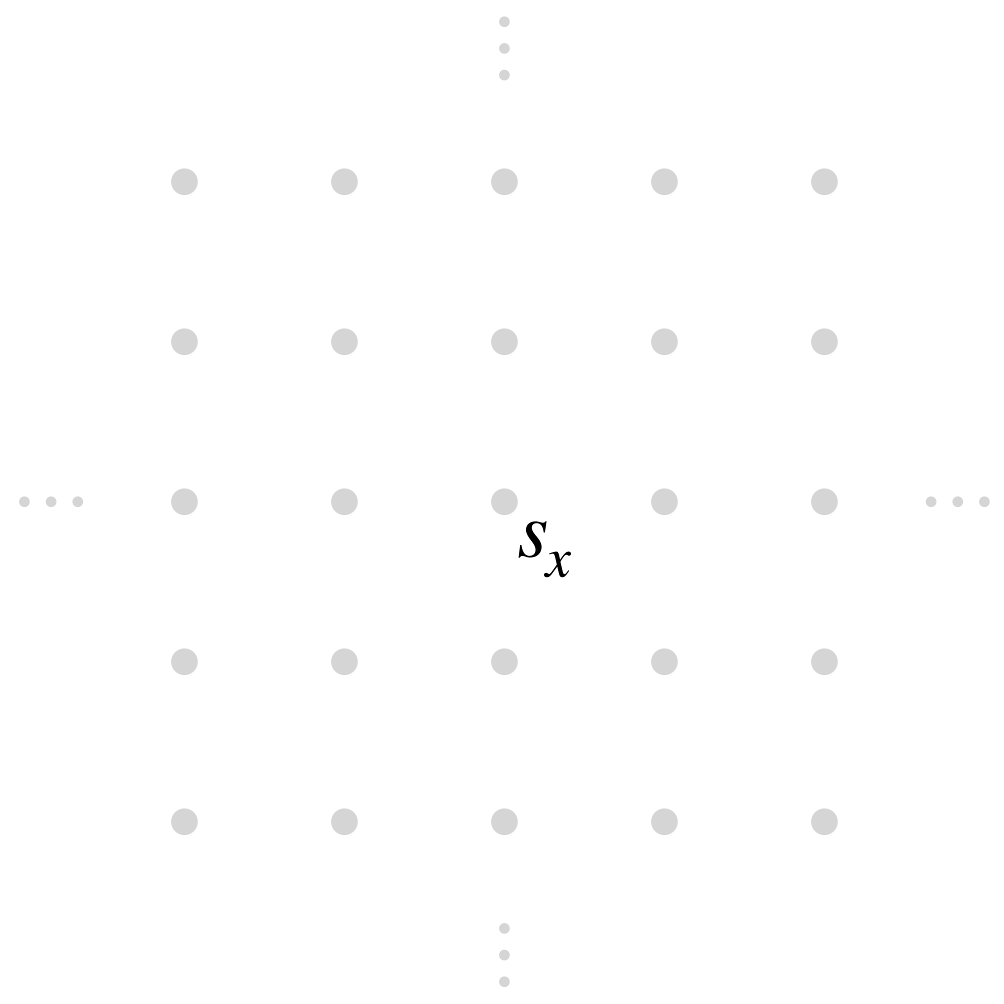
where $\langle xy \rangle$ means $x, y \in \Lambda$ such that they are directly adjacent sites

- 2nd term — external magnetic field h:
  spins want to point in the same direction as the external magnetic field (energetically favorable to do so)

- 1st term — nearest-neighbor interactions only:
  it's energetically favorable for a spin to point along the same direction as its neighbor. J is the strength of this interaction.
  $J > 0$ ferromagnetic; $J < 0$ anti-ferromagnetic

Ising Model

Physical system for intuition: magnets



Ising:

$$Z \;=\; \sum_{\substack{s_x = \pm 1 \\ x \in \Lambda}} e^{J \sum_{\langle xy \rangle} s_x s_y \,+\, h \sum_x s_x}$$

(temperature T has been absorbed into J and h)

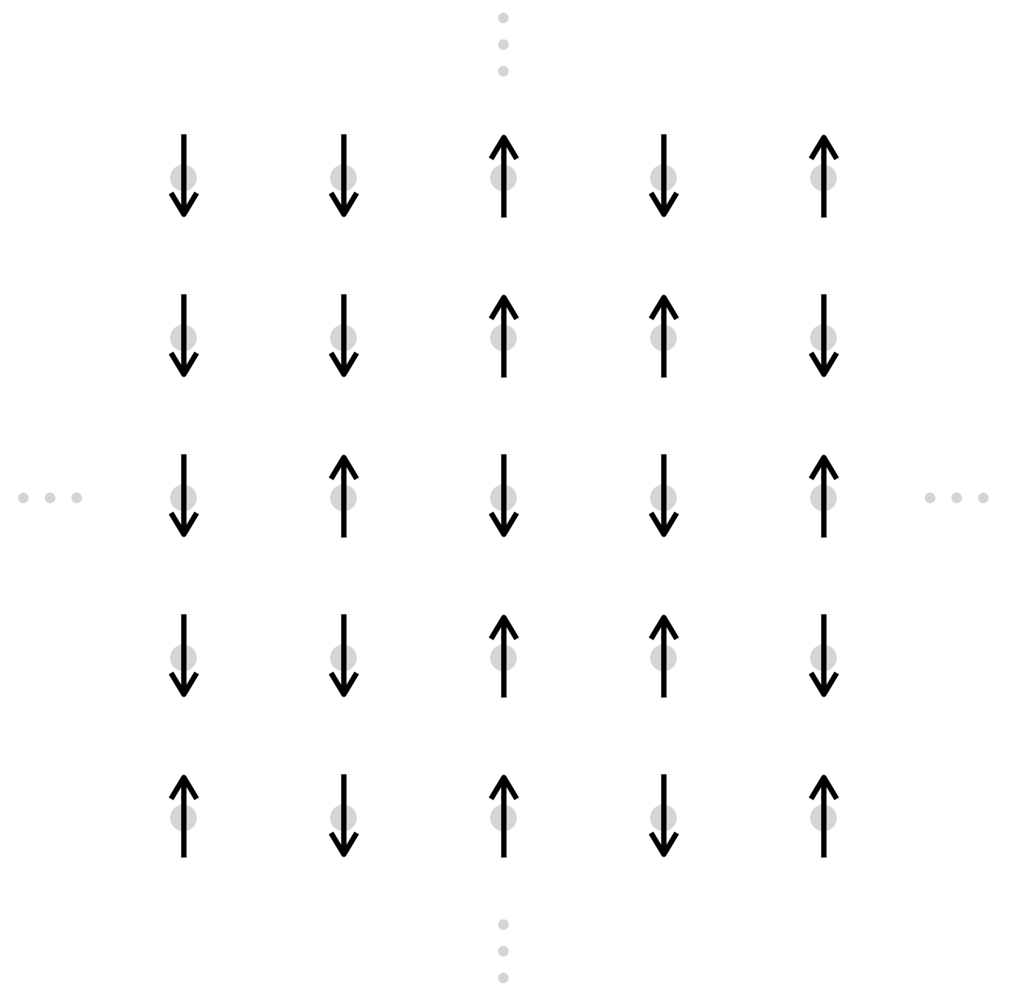
For a function $f(\{s_x\})$ of the spins,

$$\langle f(\{s_x\}) \rangle \;=\; \frac{1}{Z} \sum_{\substack{s_x = \pm 1 \\ x \in \Lambda}} f(\{s_x\})\, e^{J \sum_{\langle xy \rangle} s_x s_y \,+\, h \sum_x s_x},$$

denotes the average value of f.

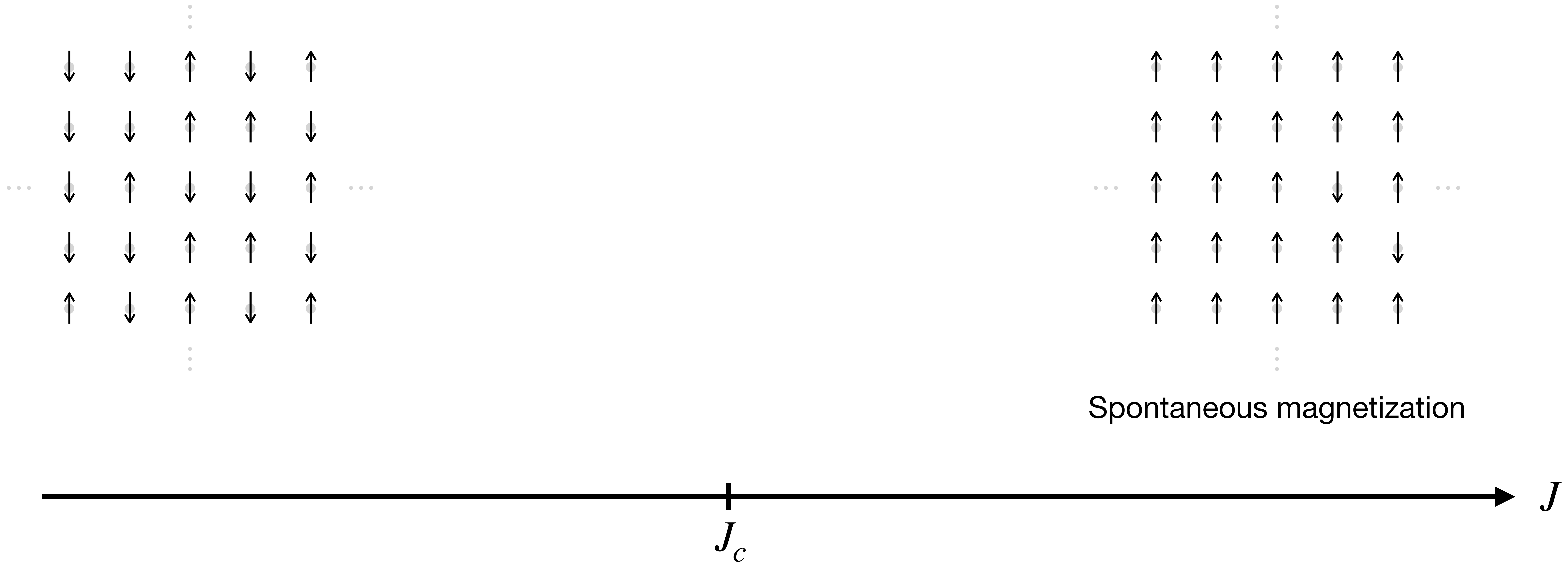
Ising Model — phase transition

Ising at $h = 0$



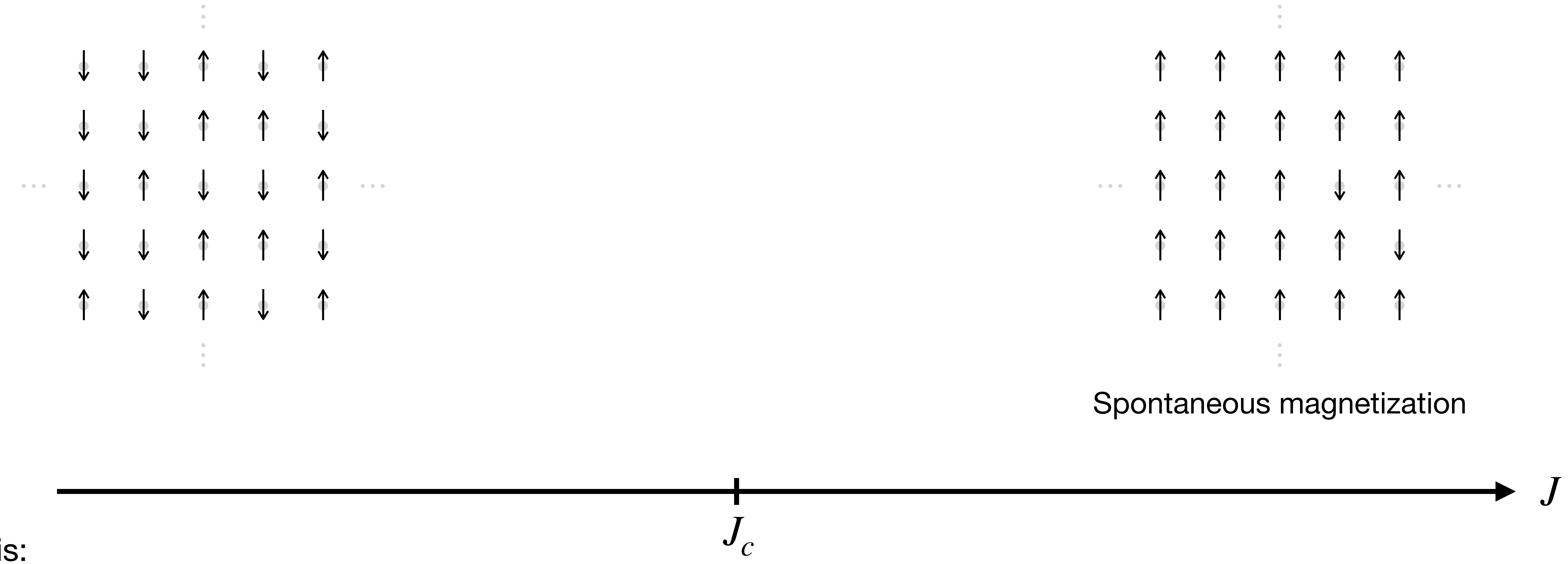
Ising Model — phase transition

Ising at $h = 0$



Ising Model — phase transition

Ising at $h = 0$



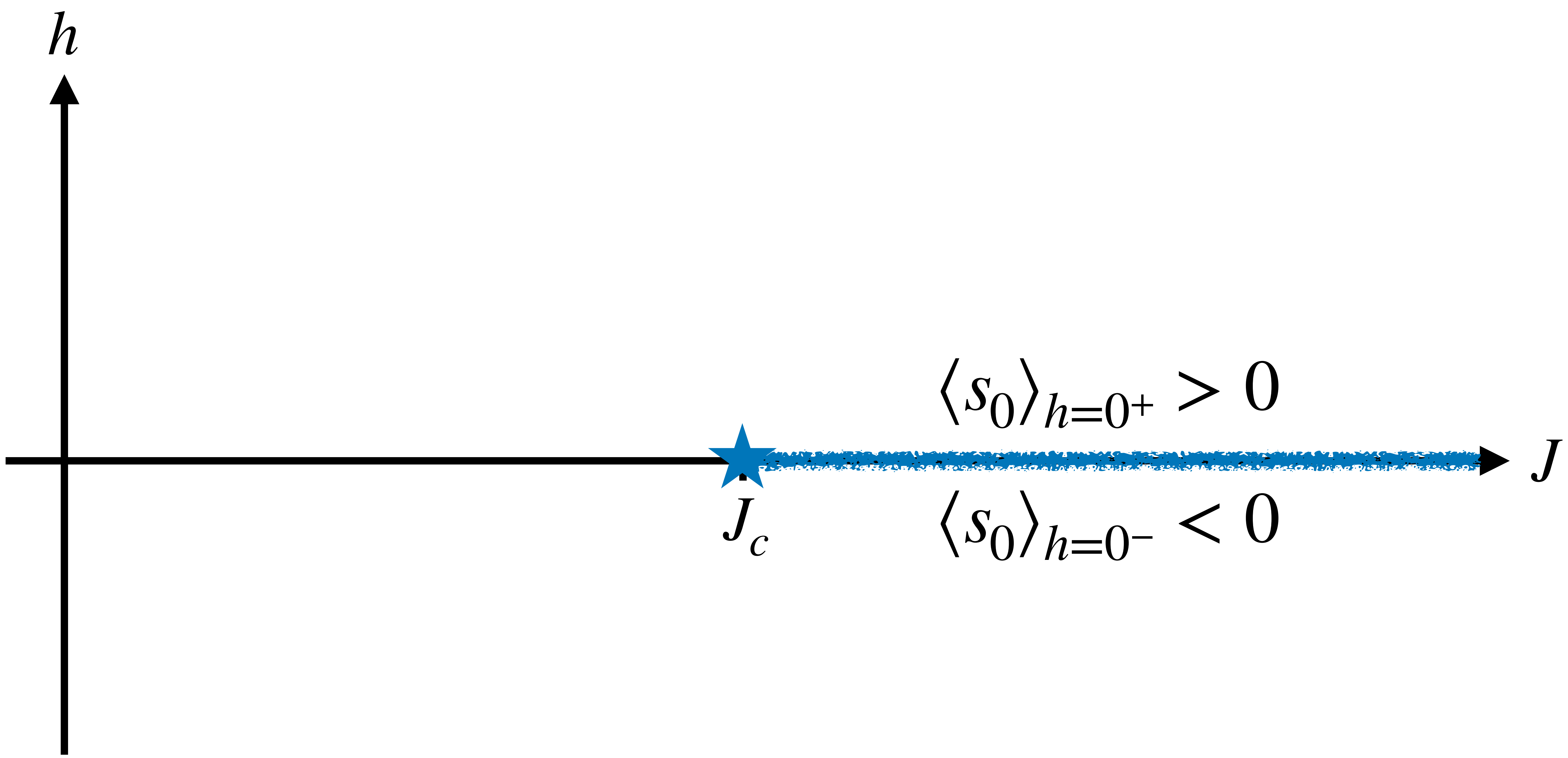
Diagnosis:
average magnetization
per site
(order parameter)

$\langle s_0 \rangle_{h=0^+} = 0$

$\langle s_0 \rangle_{h=0^+} \neq 0$

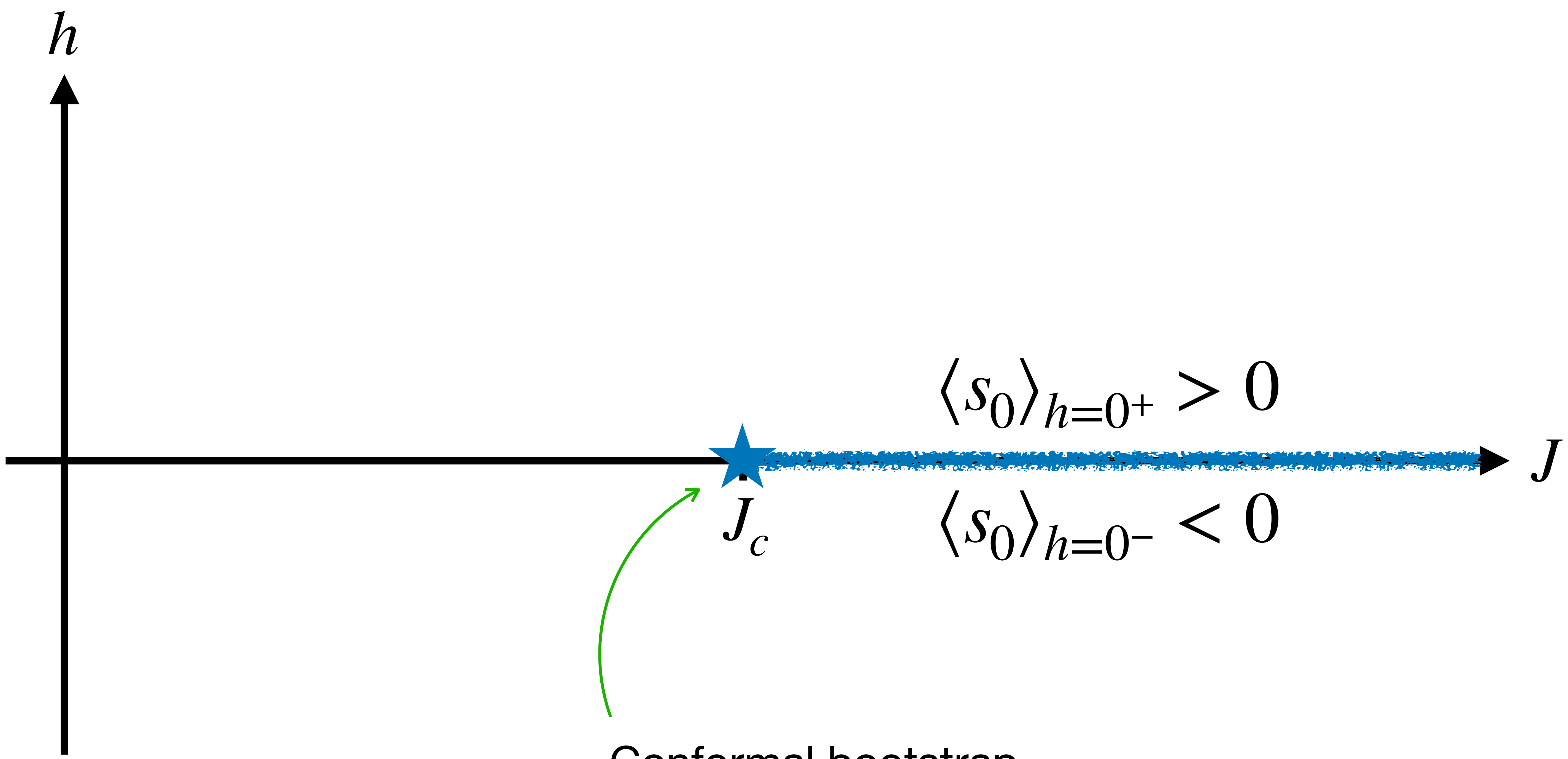
Ising Model — phase diagram

2D Ising



Ising Model — phase diagram

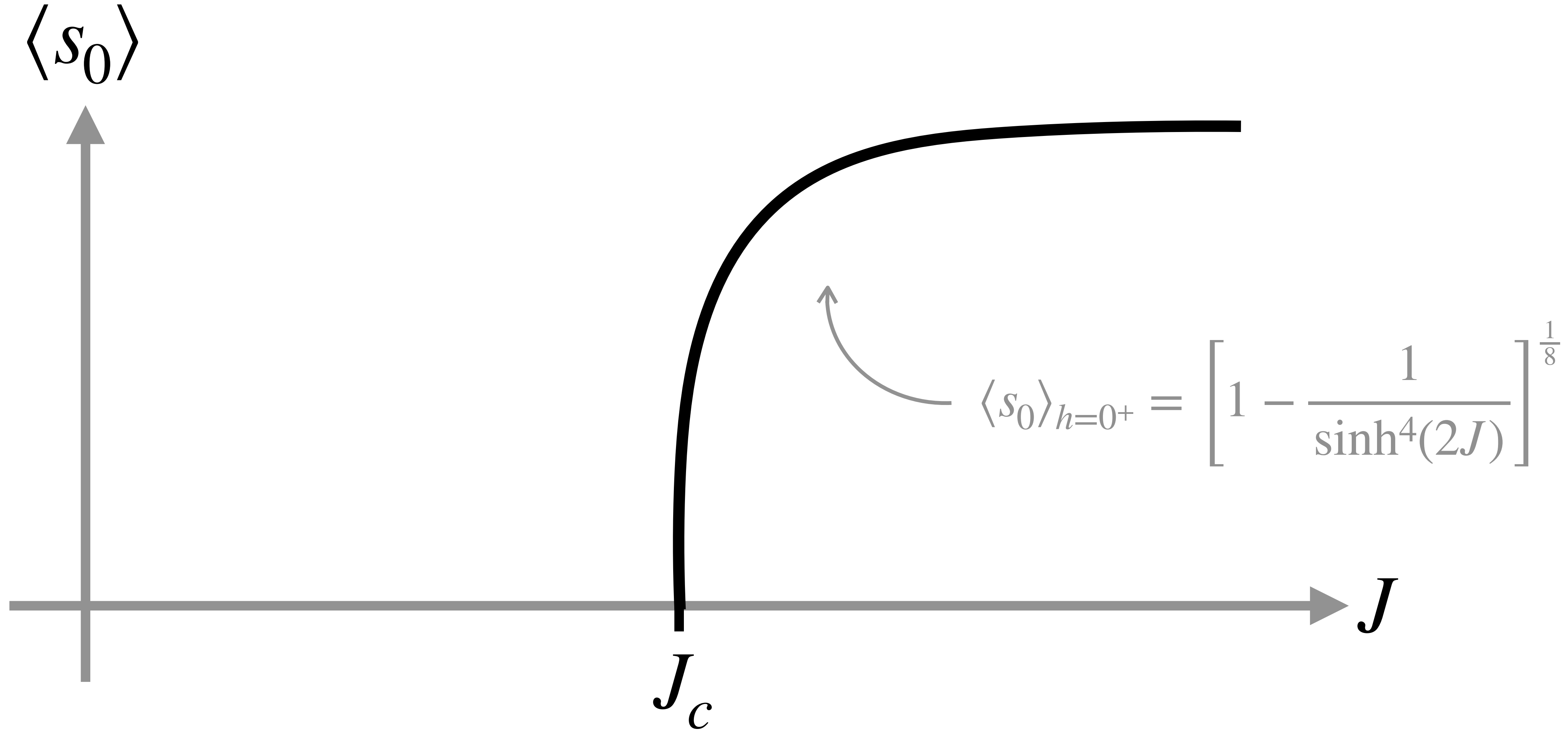
2D Ising



Conformal bootstrap
See talks by [Poland] [van Rees]

Ising Model — summary

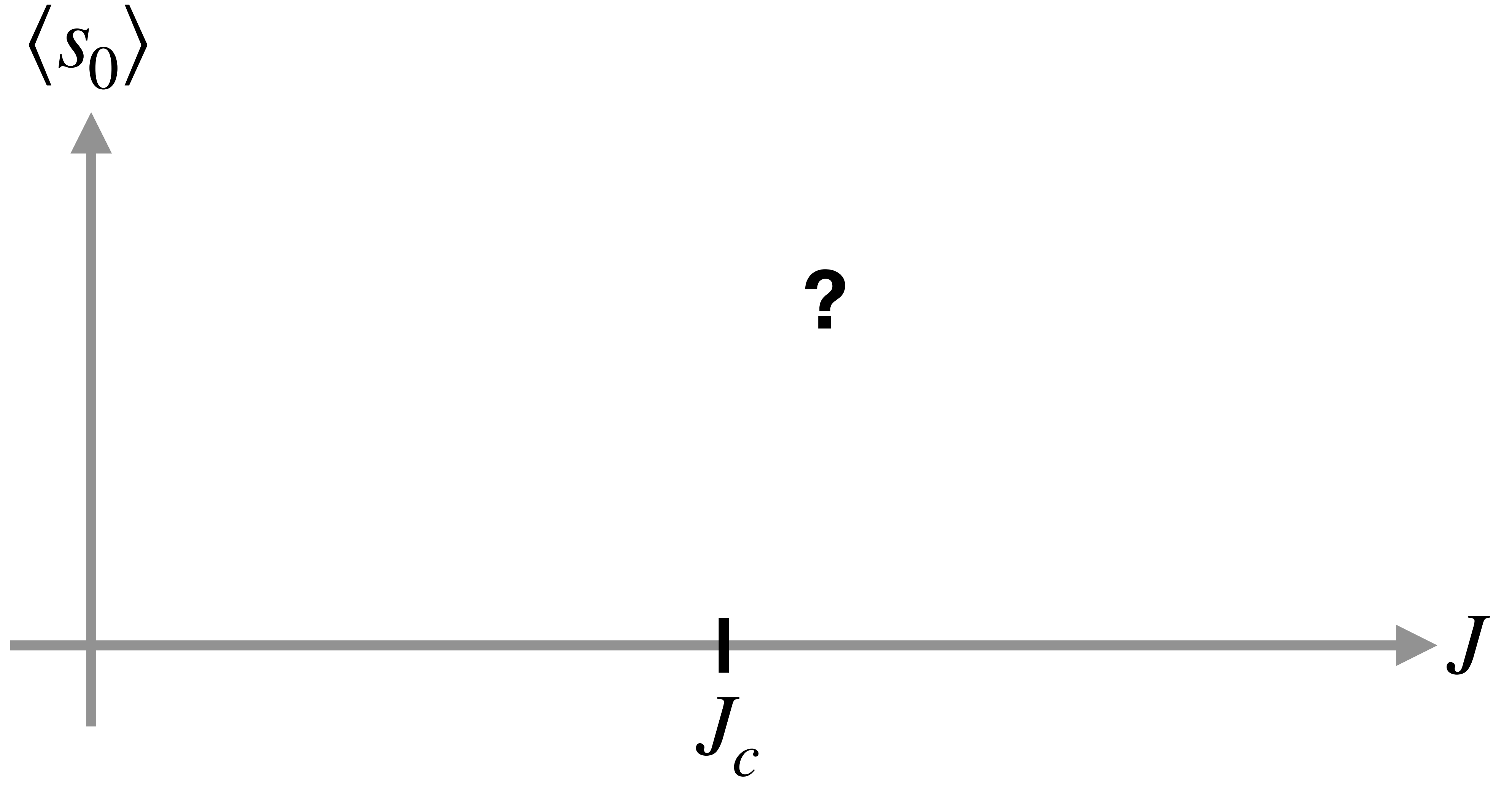
- 1D
  - Exactly soluble
  - No phase transition
- 2D
  - Exactly soluble for $h = 0$ only
  - Exhibits a phase transition!
- 3D
  - No exact solution known today
  - Exhibits a phase transition as well



Ising Model — summary

- 1D
  - Exactly soluble
  - No phase transition
- 2D
  - Exactly soluble for $h = 0$ only
  - Exhibits a phase transition!
- 3D
  - No exact solution known today
  - Exhibits a phase transition as well

Bootstrap:

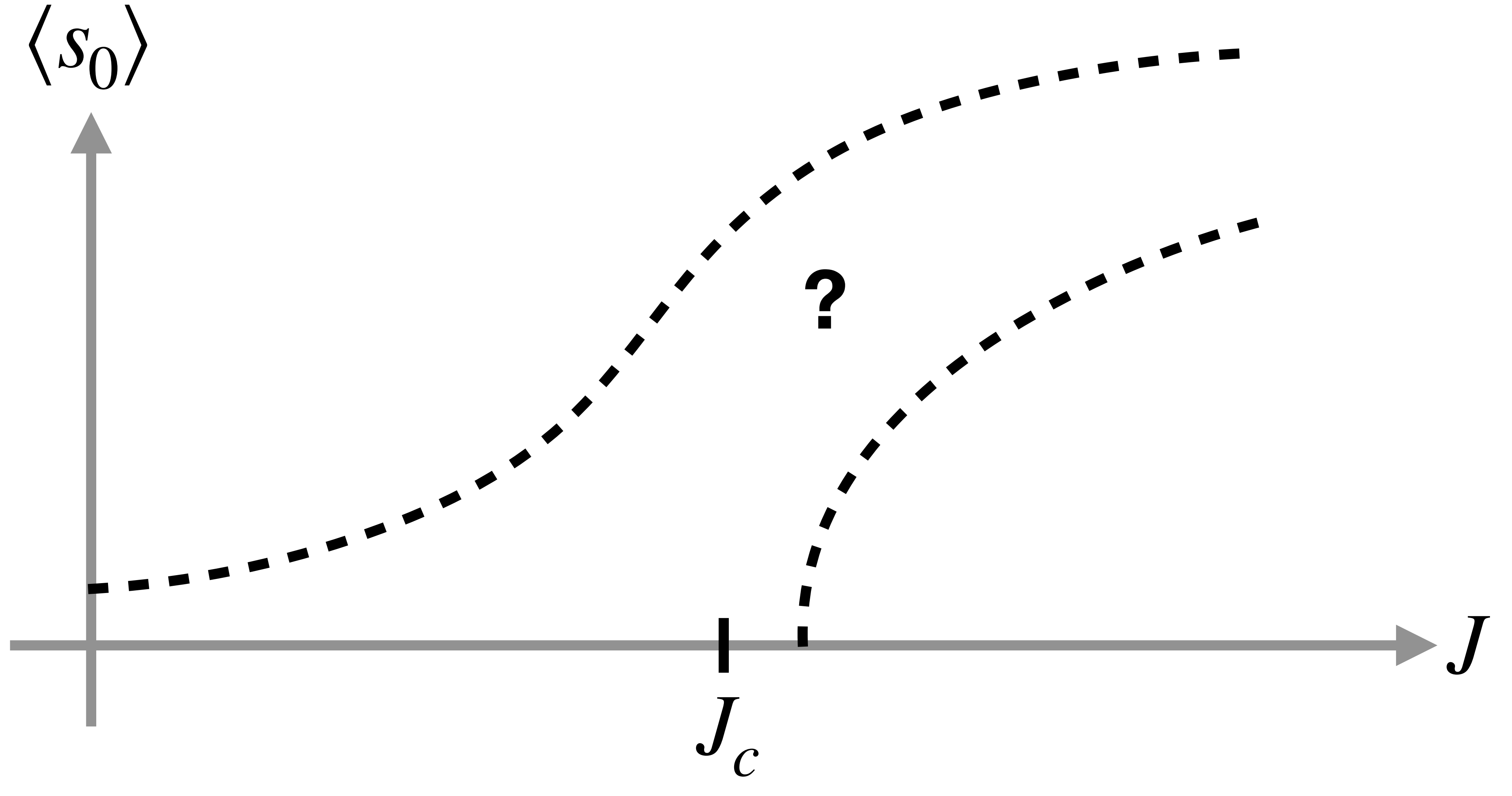


Ising Model — summary

- 1D
  - Exactly soluble
  - No phase transition
- 2D
  - Exactly soluble for $h = 0$ only
  - Exhibits a phase transition!
- 3D
  - No exact solution known today
  - Exhibits a phase transition as well

Bootstrap:

"put a bound on our ignorance"

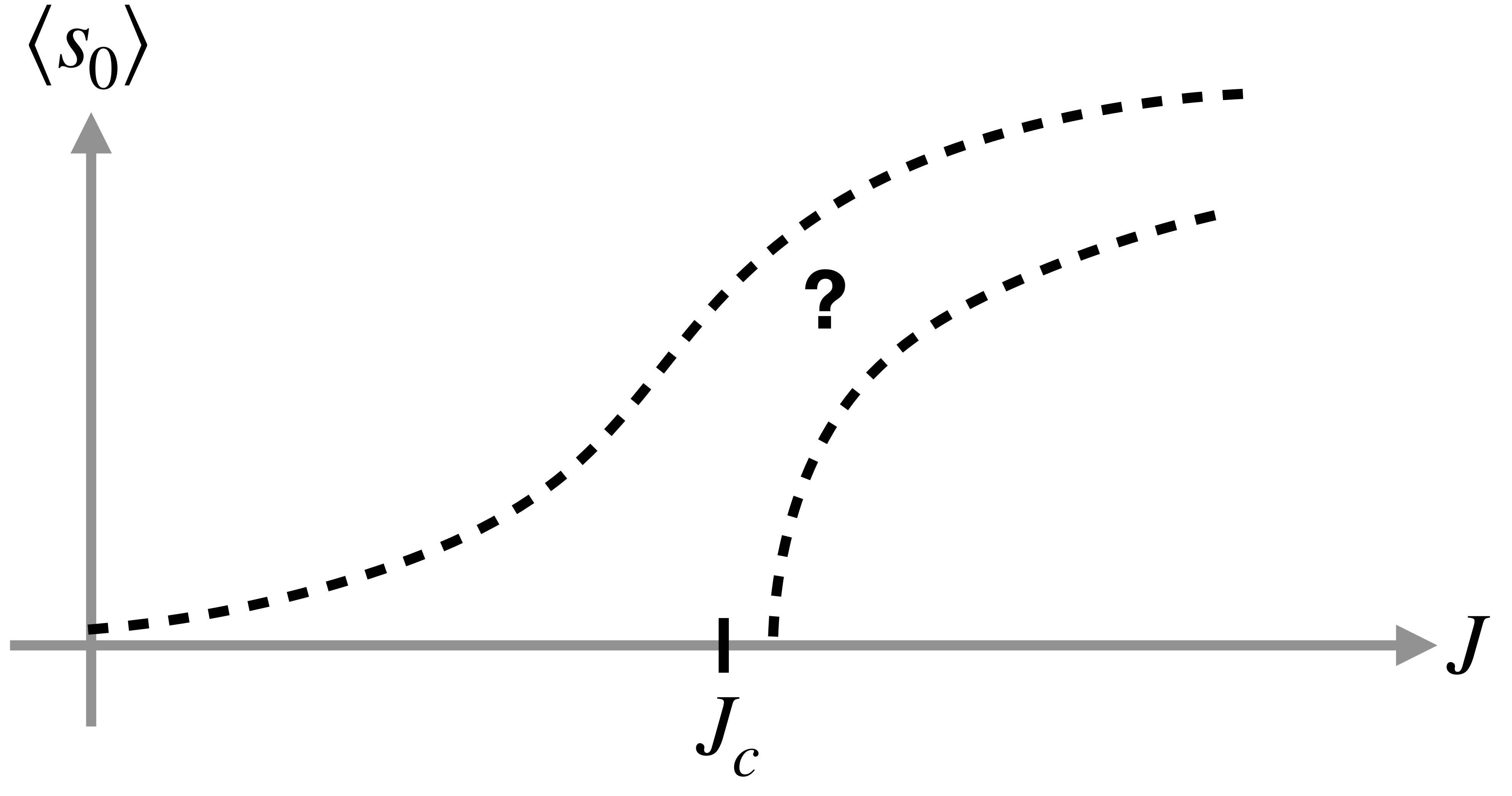


Ising Model — summary

- 1D
  - Exactly soluble
  - No phase transition
- 2D
  - Exactly soluble for $h = 0$ only
  - Exhibits a phase transition!
- 3D
  - No exact solution known today
  - Exhibits a phase transition as well

Bootstrap:

"put a bound on our ignorance"



Ising model
lattice bootstrap

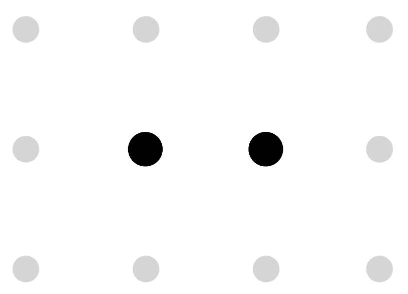
Lattice Bootstrap — Ising Model

- Objects to be bootstrapped: spin correlation functions

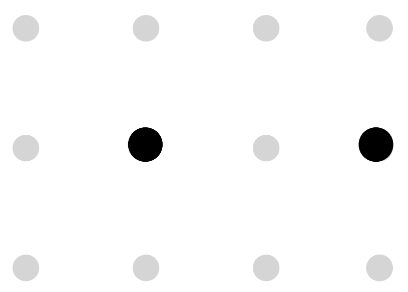
$$\langle \underline{s}_A \rangle = \frac{1}{Z} \sum_{s_x = \pm 1,\ x \in \Lambda} \underline{s}_A \, e^{J \sum_{\langle xy \rangle} s_x s_y + h \sum_x s_x}, \qquad \underline{s}_A \equiv \prod_{x \in A} s_x,$$

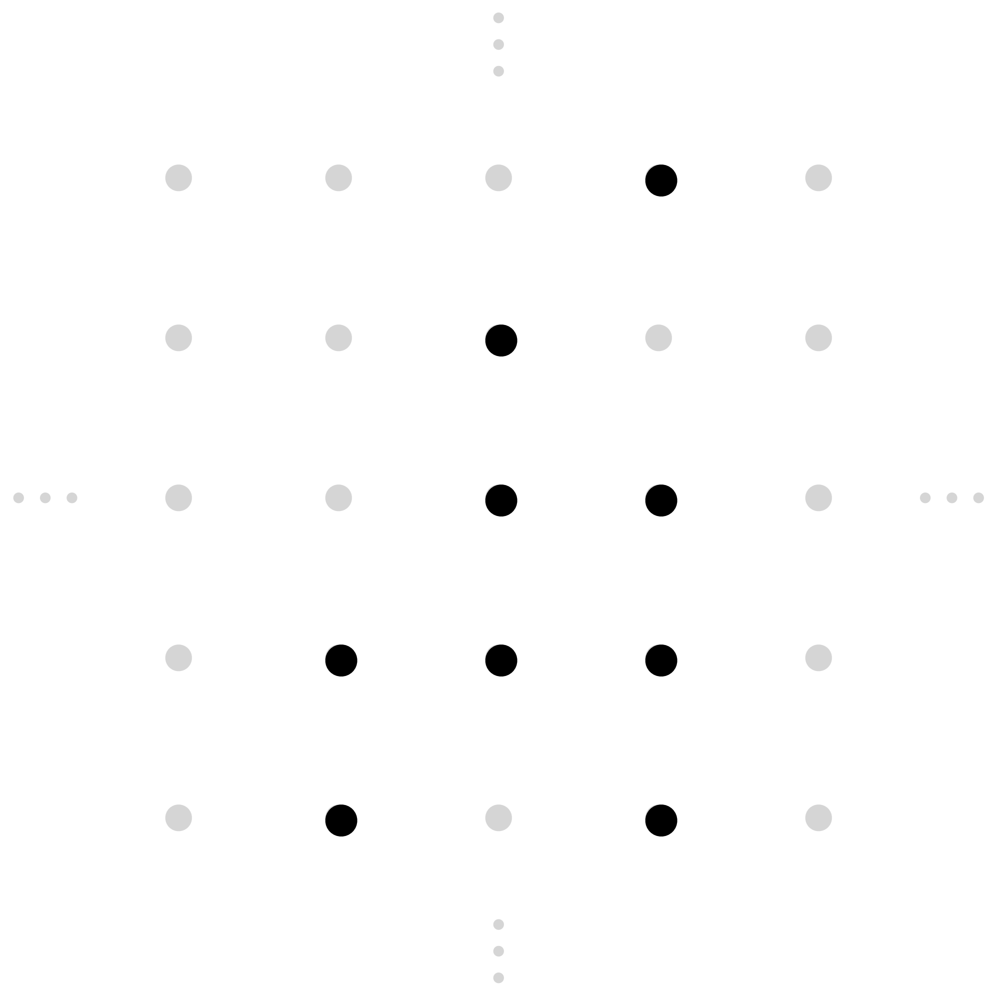
Examples:

- $\langle s_0 s_{e_1} \rangle$



- $\langle s_0 s_{2e_1} \rangle$

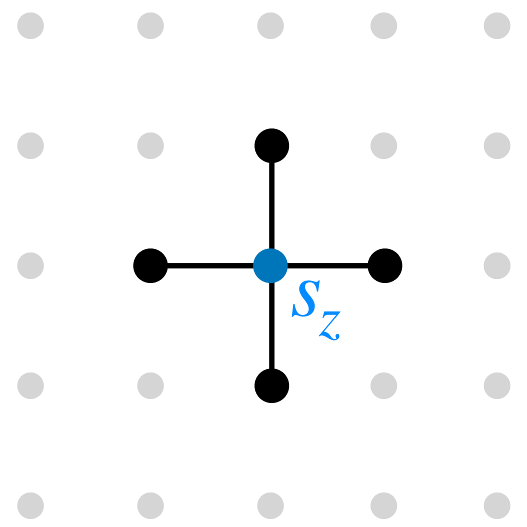




Lattice Bootstrap — Ising Model

1. Relation: spin-flip equations (from a change of variable)

$$s_z \to -s_z$$



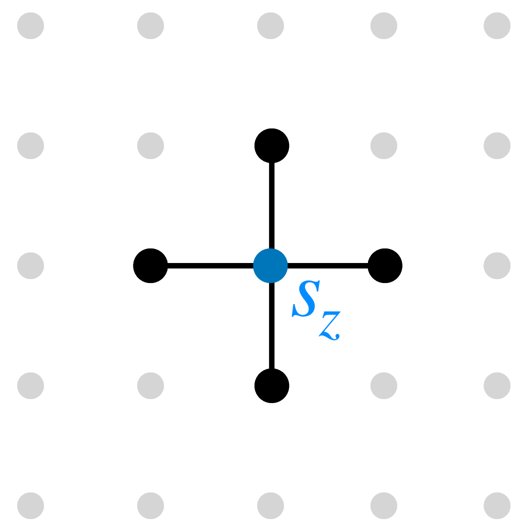
Sounds trivial, but

$$\exp\Big[-\ J\, s_z \sum_{\mu=1}^{d} (s_{z+e_\mu} + s_{z-e_\mu}) -\ h\, s_z\Big]$$

Lattice Bootstrap — Ising Model

1. Relation: spin-flip equations (from a change of variable)

$$s_z \to -s_z$$



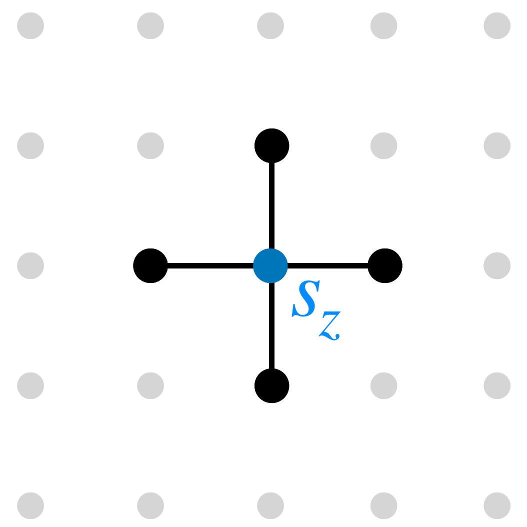
Sounds trivial, but

$$\exp\Big[-2J\,s_z\sum_{\mu=1}^{d}(s_{z+e_\mu}+s_{z-e_\mu})-2h\,s_z+J\,s_z\sum_{\mu=1}^{d}(s_{z+e_\mu}+s_{z-e_\mu})+h\,s_z\Big]$$

Lattice Bootstrap — Ising Model

1. Relation: spin-flip equations (from a change of variable)

$$s_z \to -s_z$$



Sounds trivial, but

$$\langle \underline{s}_A \rangle = \zeta_A(z) \left\langle \exp\left[-2J\, s_z \sum_{\mu=1}^{d} (s_{z+e_\mu} + s_{z-e_\mu}) - 2h\, s_z \right] \right\rangle$$

$$\zeta_A(z) = \begin{cases} -1, & \text{if } \; z \in A \\ 1, & \text{otherwise} \end{cases}$$

$$:= w \in \{0, \pm 2, \cdots, \pm d\}$$

finitely many terms

Lattice Bootstrap — Ising Model

1. Relation: spin-flip equations ($s_z = s_0$ here)

$$0 = \left[-\zeta_A(0) + \cosh(2h)\right]\langle \underline{s}_A \rangle + \sum_{\ell=0}^{2d} \left[A_\ell \cosh(2h) + B_\ell \sinh(2h)\right]\langle \underline{s}_A w^\ell \rangle$$
$$-\sinh(2h)\langle \underline{s}_A s_0 \rangle - \sum_{\ell=0}^{2d} \left[A_\ell \sinh(2h) + B_\ell \cosh(2h)\right]\langle \underline{s}_A s_0 w^\ell \rangle$$

where A_l and B_l are some fixed coefficients ($\sinh(J)$'s and $\cosh(J)$'s)

- Linear equations
- Equations between variables in a small region

Lattice Bootstrap — Ising Model

1. Relation: spin-flip equations, examples in 2D h=0

Spin correlators in the “131” diamond:

131 diamond

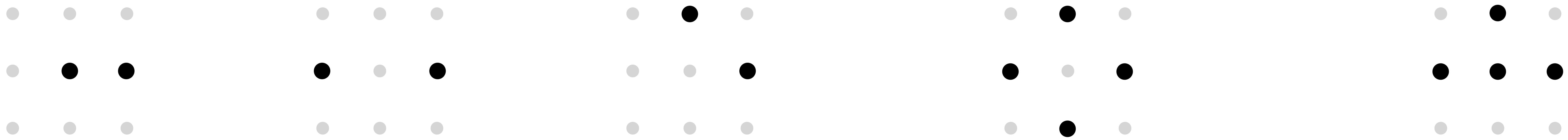
Lattice Bootstrap — Ising Model

131 diamond

1. Relation: spin-flip equations, examples in 2D h=0

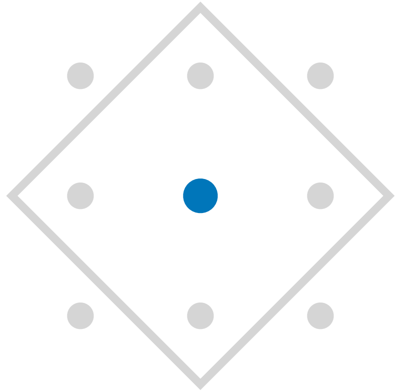
Spin correlators in the "131" diamond:

$$x_1 = \langle s_0 s_{e_1} \rangle, \quad x_2 = \langle s_{e_1} s_{-e_1} \rangle, \quad x_3 = \langle s_{e_1} s_{e_2} \rangle, \quad x_4 = \langle s_{e_1} s_{-e_1} s_{e_2} s_{-e_2} \rangle, \quad x_5 = \langle s_0 s_{e_1} s_{-e_1} s_{e_2} \rangle$$



Lattice Bootstrap — Ising Model

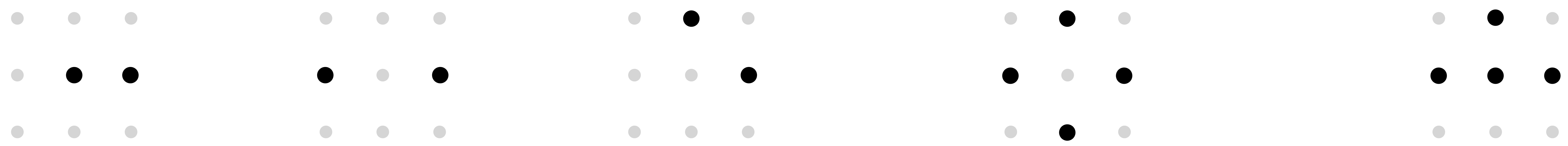
1. Relation: spin-flip equations, examples in 2D h=0



131 diamond

Spin correlators in the "131" diamond:

$$x_1 = \langle s_0 s_{e_1} \rangle, \quad x_2 = \langle s_{e_1} s_{-e_1} \rangle, \quad x_3 = \langle s_{e_1} s_{e_2} \rangle, \quad x_4 = \langle s_{e_1} s_{-e_1} s_{e_2} s_{-e_2} \rangle, \quad x_5 = \langle s_0 s_{e_1} s_{-e_1} s_{e_2} \rangle$$



Spin-flip equations relates spin correlators:
(total of 6 spin-flip eqs for 131, not all independent)

$$(A = \varnothing) \qquad 0 = A_2\left(4 + 4x_2 + 8x_3\right) + A_4\left(40 + 64x_2 + 128x_3 + 24x_4\right) - 4B_1x_1 - B_3\left(40x_1 + 24x_5\right)$$

$$A_2 = \frac{-15 + 16\cosh(4J) - \cosh(8J)}{48}$$
$$A_4 = \frac{3 - 4\cosh(4J) + \cosh(8J)}{192}$$
$$B_1 = \frac{8\sinh(4J) - \sinh(8J)}{12}$$
$$B_3 = \frac{-2\sinh(4J) + \sinh(8J)}{48}$$

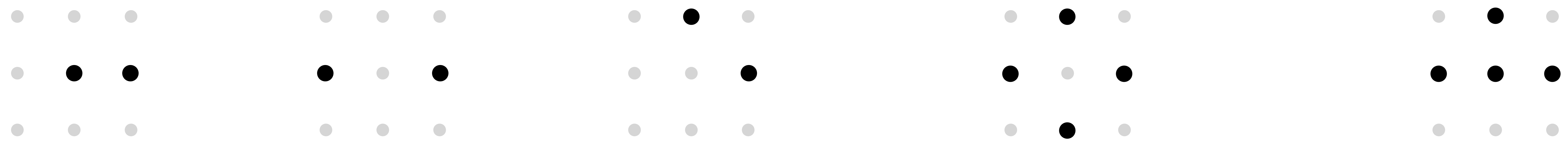
Lattice Bootstrap — Ising Model

1. Relation: spin-flip equations, examples in 2D h=0

131 diamond

Spin correlators in the "131" diamond:

$$x_1 = \langle s_0 s_{e_1} \rangle, \quad x_2 = \langle s_{e_1} s_{-e_1} \rangle, \quad x_3 = \langle s_{e_1} s_{e_2} \rangle, \quad x_4 = \langle s_{e_1} s_{-e_1} s_{e_2} s_{-e_2} \rangle, \quad x_5 = \langle s_0 s_{e_1} s_{-e_1} s_{e_2} \rangle$$



Spin-flip equations relates spin correlators:
(total of 6 spin-flip eqs for 131, not all independent)

$$(A = \varnothing) \quad 0 = A_2 \left(4 + 4x_2 + 8x_3\right) + A_4 \left(40 + 64x_2 + 128x_3 + 24x_4\right) - 4B_1 x_1 - B_3 \left(40x_1 + 24x_5\right)$$

$$A_2 = \frac{-15 + 16\cosh(4J) - \cosh(8J)}{48}$$
$$A_4 = \frac{3 - 4\cosh(4J) + \cosh(8J)}{192}$$
$$B_1 = \frac{8\sinh(4J) - \sinh(8J)}{12}$$
$$B_3 = \frac{-2\sinh(4J) + \sinh(8J)}{48}$$

Correlators x_4 and x_5 are not independent:

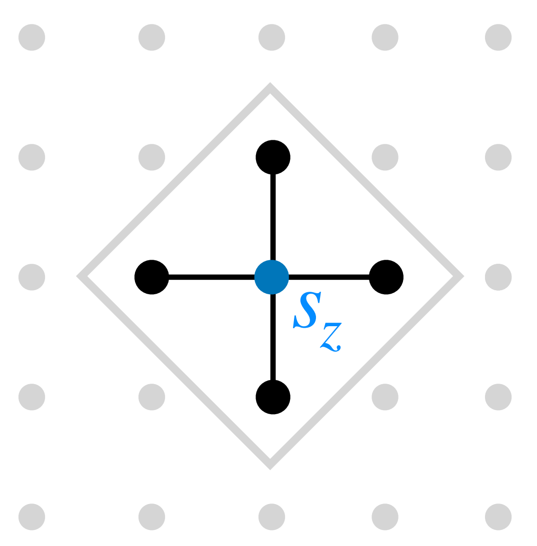
$$x_4 = \frac{-8(\cosh(2J) + \cosh(6J))x_1 + \sinh(2J)(-1 + 2x_2 + 4x_3) + \sinh(6J)(3 + 2x_2 + 4x_3)}{4\sinh^3(2J)}$$

$$x_5 = \frac{-(1 + 3\cosh(4J))x_1 + \sinh(4J)(1 + x_2 + 2x_3)}{2\sinh^2(2J)}$$

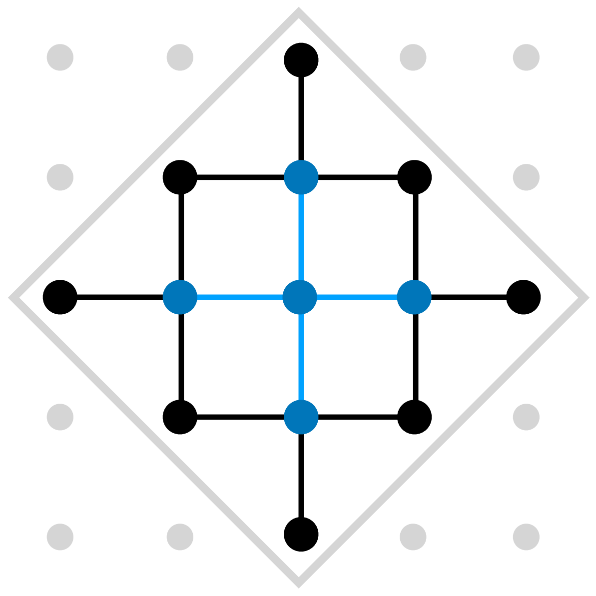
Lattice Bootstrap — Ising Model

1. Relation: spin-flip equations

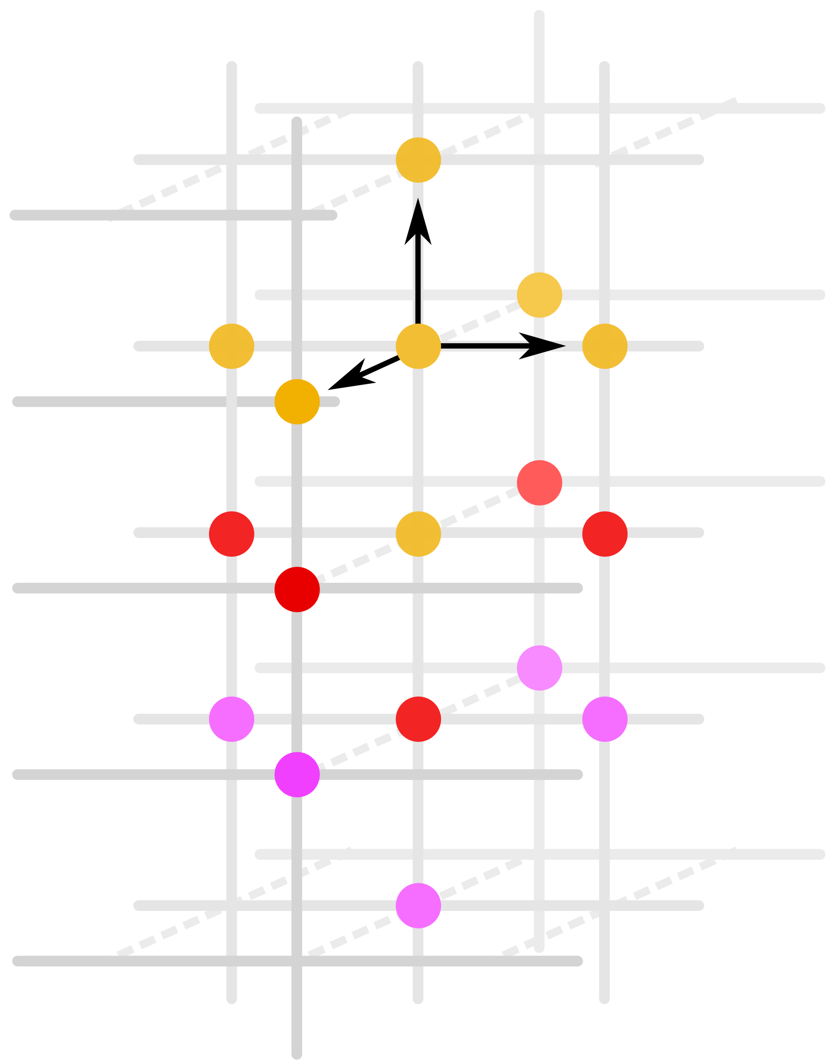
- Linear equations
- Equations between variables in a small subregion



131 diamond



13531 diamond



15551 domain

Lattice Bootstrap — Ising Model

1. Relation: spin-flip equations

	primite subsets	ind. spin-flip equations	ind. spin correlators
2D 131, h=0	6	2	3
2D 13531, h=0	569	549	19
2D 13531, h≠0	1127	1097	29
3D 15551, h=0	5214	4584	629

- Solve numerically
- Not the bottleneck of the computation

Lattice Bootstrap — Ising Model

2. Positivity: several kinds

- Reflection positivity
- Square positivity (appears to be redundant)
- Griffiths inequalities

Lattice Bootstrap — Ising Model

2. Reflection Positivity

$$\langle \mathcal{O}^R \, \mathcal{O} \rangle \geq 0 \,, \qquad \text{where} \qquad \mathcal{O} = \sum_{A \subset H} t^A \underline{s}_A \,, \qquad \mathcal{O}^R = \sum_{A \subset H} t^A \underline{s}_{R(A)}$$

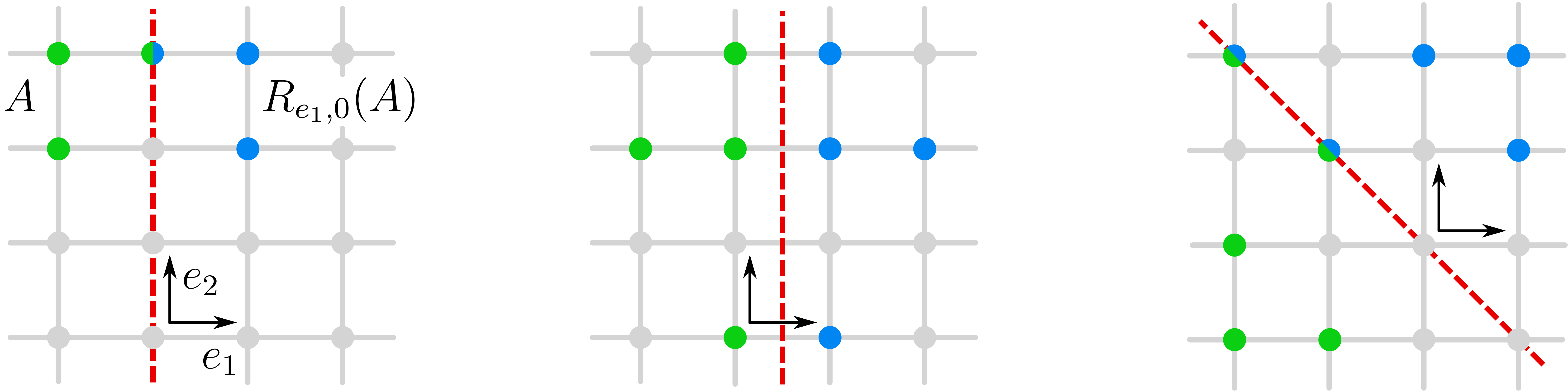
The three inequivalent reflection planes: $R_{v,c}(x) = x - \frac{2(v \cdot x - c)}{v^2} v \qquad H = \{x \in \Lambda : v \cdot x \geq c\}$

Lattice Bootstrap — Ising Model

2. Reflection Positivity

$$\langle \mathcal{O}^R \mathcal{O} \rangle \geq 0\,, \qquad \text{where} \quad \mathcal{O} = \sum_{A \subset H} t^A \underline{s}_A\,, \quad \mathcal{O}^R = \sum_{A \subset H} t^A \underline{s}_{R(A)}$$

The three inequivalent reflection planes: $R_{v,c}(x) = x - \frac{2(v \cdot x - c)}{v^2} v$ $\quad H = \{x \in \Lambda : v \cdot x \geq c\}$



$R_{e_1,0}$ $\qquad R_{e_1,\frac{1}{2}}$ $(J \geq 0)$ $\qquad R_{e_1+e_2,0}$

Lattice Bootstrap — Ising Model

2. Reflection Positivity

$$\langle \mathcal{O}^R \, \mathcal{O} \rangle \geq 0 \,, \qquad \text{where} \quad \mathcal{O} = \sum_{A \subset H} t^A \underline{s}_A \,, \quad \mathcal{O}^R = \sum_{A \subset H} t^A \underline{s}_{R(A)}$$

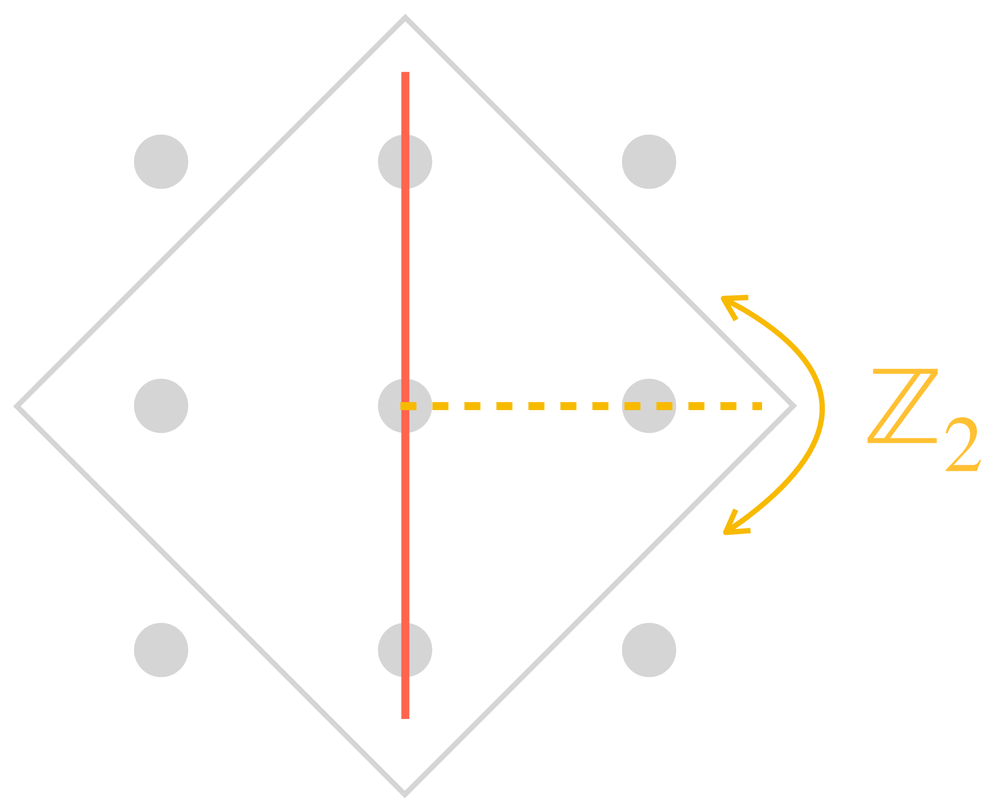
Equivalently, $\vec{t}^{\,T} M \vec{t} \geq 0$ with $M_{AA'} := \langle s_{R(A)} s_{A'} \rangle \;\; \Leftrightarrow \;\; M \succeq 0$

Lattice Bootstrap — Ising Model

2. Reflection Positivity, example 131 diamond

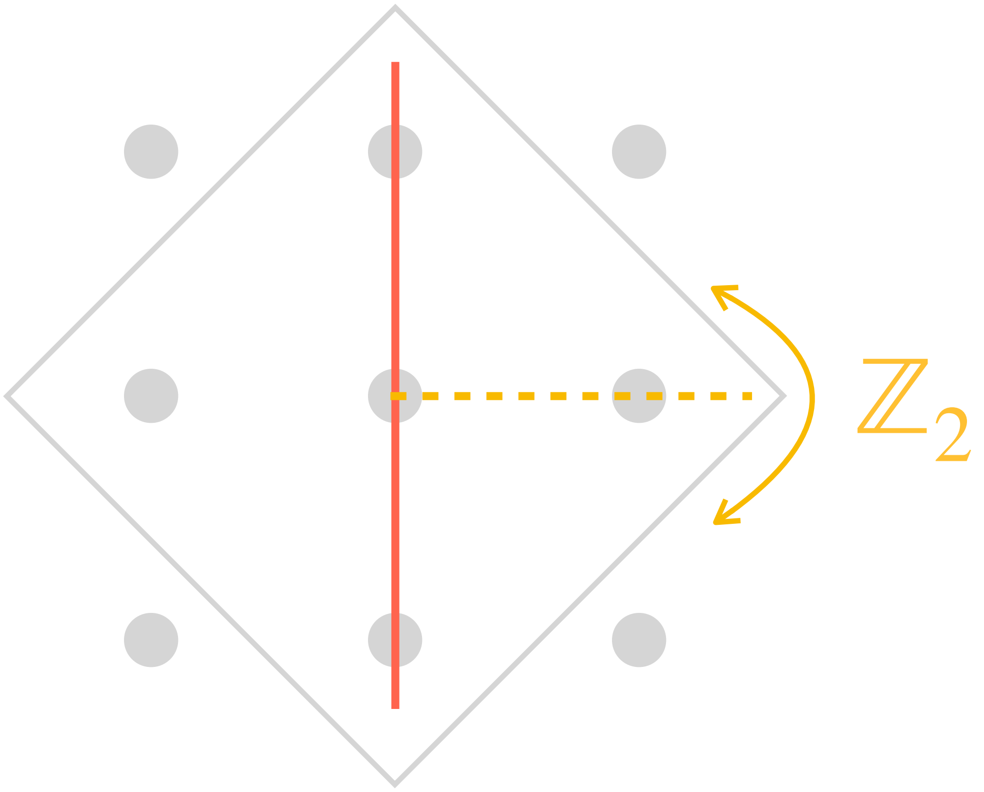
Lattice Bootstrap — Ising Model

2. Reflection Positivity, example 131 diamond



Lattice Bootstrap — Ising Model

2. Reflection Positivity, example 131 diamond



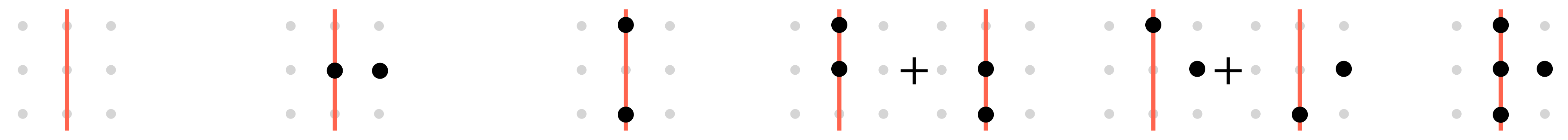
$$\text{even}: \; 1,\; s_0 s_{e_1},\; s_{e_2} s_{-e_2},\; \frac{s_0 s_{e_2} + s_0 s_{-e_2}}{2},\; \frac{s_{e_1} s_{e_2} + s_{e_1} s_{-e_2}}{2},\; s_0 s_{e_1} s_{e_2} s_{-e_2};$$

$$\text{odd}: \; \frac{s_0 s_{e_2} - s_0 s_{-e_2}}{2},\; \frac{s_{e_1} s_{e_2} - s_{e_1} s_{-e_2}}{2}.$$

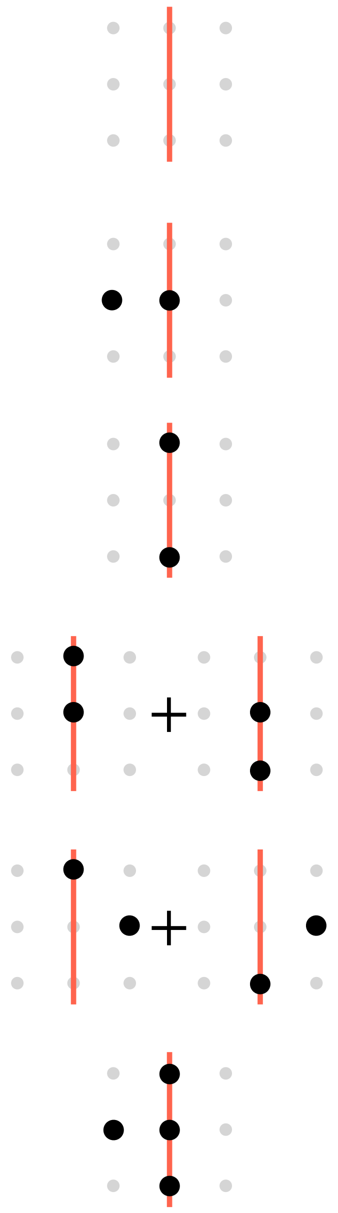
Invariant SDP: sufficient to impose positive semidefiniteness of matrices built from states that transform in each irrep of the symmetry group.

Lattice Bootstrap — Ising Model

2. Reflection Positivity, example



s_A



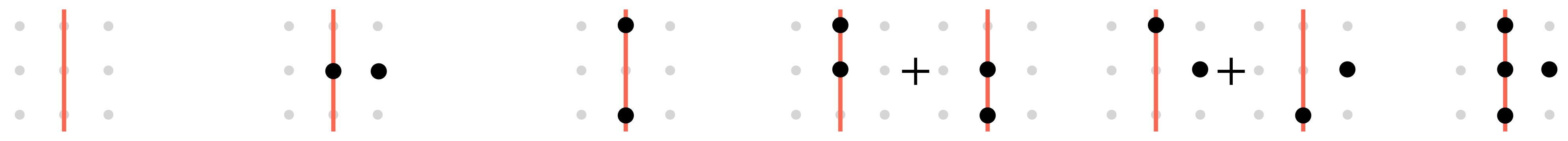
$s_{R(A)}$

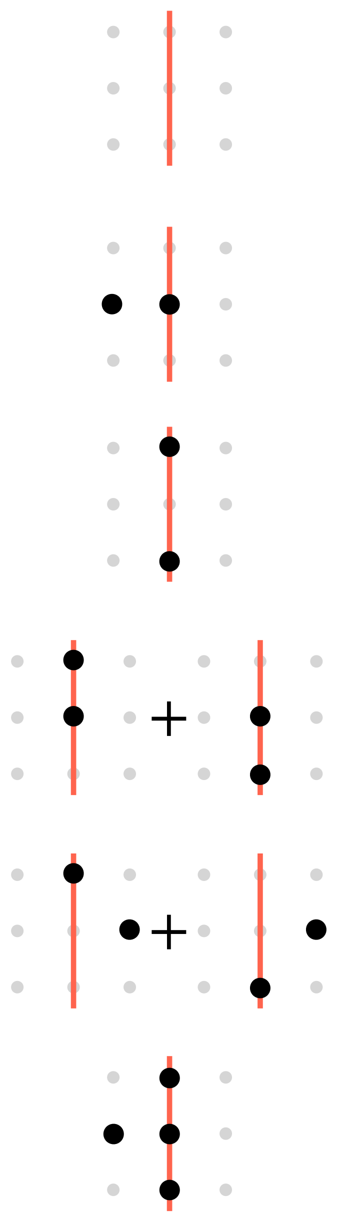
$$\begin{pmatrix} 1 & x_1 & x_2 & x_1 & x_3 & x_5 \\ x_1 & x_2 & x_5 & x_3 & x_5 & x_4 \\ x_2 & x_5 & 1 & x_1 & x_3 & x_1 \\ x_1 & x_3 & x_1 & \frac{1+x_2}{2} & \frac{x_1+x_5}{2} & x_3 \\ x_3 & x_5 & x_3 & \frac{x_1+x_5}{2} & \frac{x_2+x_4}{2} & x_5 \\ x_5 & x_4 & x_1 & x_3 & x_5 & x_2 \end{pmatrix} \succeq 0$$

$x_1 = \langle s_0 s_{e_1} \rangle, \quad x_2 = \langle s_{e_1} s_{-e_1} \rangle, \quad x_3 = \langle s_{e_1} s_{e_2} \rangle, \quad x_4 = \langle s_{e_1} s_{-e_1} s_{e_2} s_{-e_2} \rangle, \quad x_5 = \langle s_0 s_{e_1} s_{-e_1} s_{e_2} \rangle$

Lattice Bootstrap — Ising Model

2. Reflection Positivity, example





$$\begin{pmatrix} 1 & x_1 & x_2 & x_1 & x_3 & x_5 \\ x_1 & x_2 & x_5 & x_3 & x_5 & x_4 \\ x_2 & x_5 & 1 & x_1 & x_3 & x_1 \\ x_1 & x_3 & x_1 & \frac{1+x_2}{2} & \frac{x_1+x_5}{2} & x_3 \\ x_3 & x_5 & x_3 & \frac{x_1+x_5}{2} & \frac{x_2+x_4}{2} & x_5 \\ x_5 & x_4 & x_1 & x_3 & x_5 & x_2 \end{pmatrix} \succeq 0$$

$x_1 = \langle s_0 s_{e_1} \rangle, \quad x_2 = \langle s_{e_1} s_{-e_1} \rangle, \quad x_3 = \langle s_{e_1} s_{e_2} \rangle, \quad x_4 = \langle s_{e_1} s_{-e_1} s_{e_2} s_{-e_2} \rangle, \quad x_5 = \langle s_0 s_{e_1} s_{-e_1} s_{e_2} \rangle$

Lattice Bootstrap — Ising Model

2. Griffith's inequalities (Positivity) [Glimm-Jaffe]

$$\langle \underline{s}_A \rangle \geq 0 \qquad (G_1)$$

$$\langle \underline{s}_A \underline{s}_B \rangle - \langle \underline{s}_A \rangle \langle \underline{s}_B \rangle \geq 0 \qquad (G_2)$$

for finite subsets $A, B \subset \Lambda$.

Lattice Bootstrap — Ising Model

2. Griffith's inequalities (Positivity) [Glimm-Jaffe]

$$\langle \underline{s}_A \rangle \geq 0 \qquad (G_1)$$

$$\langle \underline{s}_A \underline{s}_B \rangle - \langle \underline{s}_A \rangle \langle \underline{s}_B \rangle \geq 0 \qquad (G_2)$$

for finite subsets $A, B \subset \Lambda$.

- True for ferromagnetic coupling $J \geq 0$.
- G_2 implies $\langle \underline{s}_A \rangle$ are monotonic as functions of J or h.
- G_2 are non-linear inequalities. Many of them are non-convex.
  Thus far, we have not been able to implement them in SDP in a useful way. More on this later.

Lattice Bootstrap — Ising Model

SDP problem:

- Reflection positivity matrices, one for each irrep of symmetry group:

  $$X^{(k)} = \sum_{A \subset \mathcal{D}} Y_A^{(k)} \langle \underline{s}_A \rangle \succeq 0, \ \forall k$$

  (e.g. k={even, odd} in previous slide)

- Plug-in numerical solution of spin-flip equations $\langle \underline{s}_A \rangle = \sum_I a_A^I \langle \underline{s}_I \rangle + c_A$, where $\langle \underline{s}_I \rangle$ are the independent variables, and so

  $$X^{(k)} = \sum_I W_I^{(k)} \langle \underline{s}_I \rangle + V^{(k)} \succeq 0, \ \forall k$$

  where $W_I^{(k)} = \sum_A a_A^I Y_A^{(k)}$, $V^{(k)} = \sum_A c_A Y_A^{(k)}$

Lattice Bootstrap — Ising Model

SDP problem:

- Reflection positivity matrices, one for each irrep of symmetry group:

  $$X^{(k)} = \sum_{A\subset\mathscr{D}} Y_A^{(k)} \langle \underline{s}_A \rangle \succeq 0,\ \forall k$$

  (e.g. k={even, odd} in previous slide)

- Plug-in numerical solution of spin-flip equations $\langle \underline{s}_A \rangle = \sum_I a_A^I \langle \underline{s}_I \rangle + c_A$, where $\langle \underline{s}_I \rangle$ are the independent variables, and so

  $$X^{(k)} = \sum_I W_I^{(k)} \langle \underline{s}_I \rangle + V^{(k)} \succeq 0,\ \forall k$$

  where $W_I^{(k)} = \sum_A a_A^I Y_A^{(k)}$, $V^{(k)} = \sum_A c_A Y_A^{(k)}$

$$\min_{y_I\in\mathbb{R}} \sum_I b^I y_I$$

subject to $\displaystyle\sum_I a_A^I y_I + c_A \geq 0,\quad \forall A \quad (G_1)$

and $\displaystyle\sum_I W_I^{(k)} y_I + V^{(k)} \succeq 0,\quad \forall k \quad (RP)$

Solve using MOSEK or SDPA-QD.

Did not impose G_2

Lattice Bootstrap — Ising Model

Some numbers:

<u>2D 13531 diamond h≠0</u>

- 29 independent variables
- 8 positive semidefinite matrices $(288^2, 224^2, 12^2, 4^2, 144^2, 112^2, 20^2, 12^2)$

<u>3D 15551 domain h=0</u>

- 629 independent variables
- 17 positive semidefinite matrices
  Largest matrix: 2400×2400
- Too big for SDP solver. Had to truncate matrices to 100×100

Lattice Bootstrap — Ising Model

Some numbers:

<u>2D 13531 diamond h≠0</u>

- 29 independent variables
- 8 positive semidefinite matrices $(288^2, 224^2, 12^2, 4^2, 144^2, 112^2, 20^2, 12^2)$

<u>3D 15551 domain h=0</u>

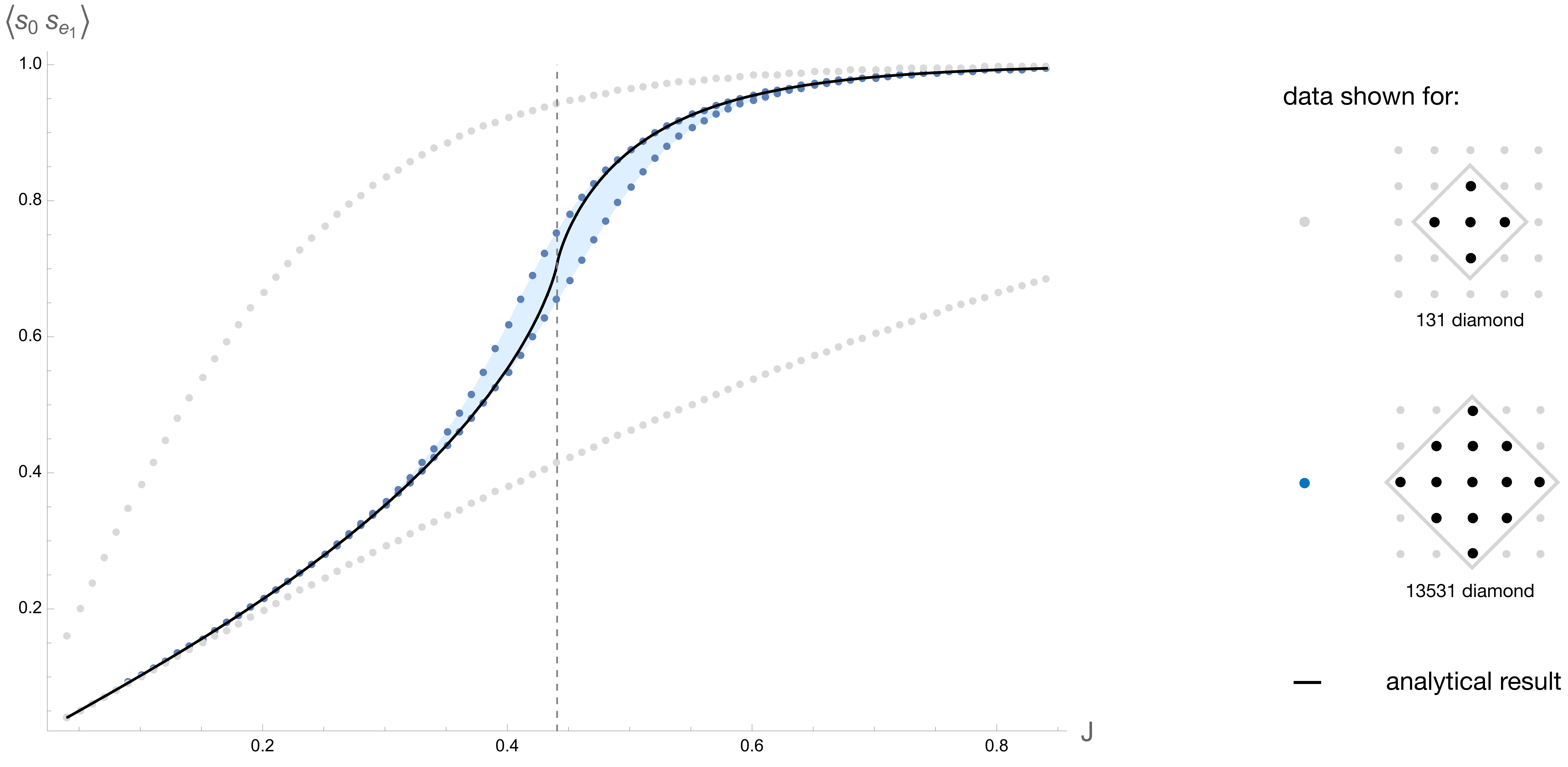
- 629 independent variables
- 17 positive semidefinite matrices Largest matrix: 2400×2400
- Too big for SDP solver. Had to truncate matrices to 100×100

Large scale separation in positive-semidefinite matrices $\sim 10^{10}$

- Effectively lose 10 digits of accuracy
- SDPA-QD for most precise results, and MOSEK for when ≤ 6 digits is enough (which is a much faster SDP solver).

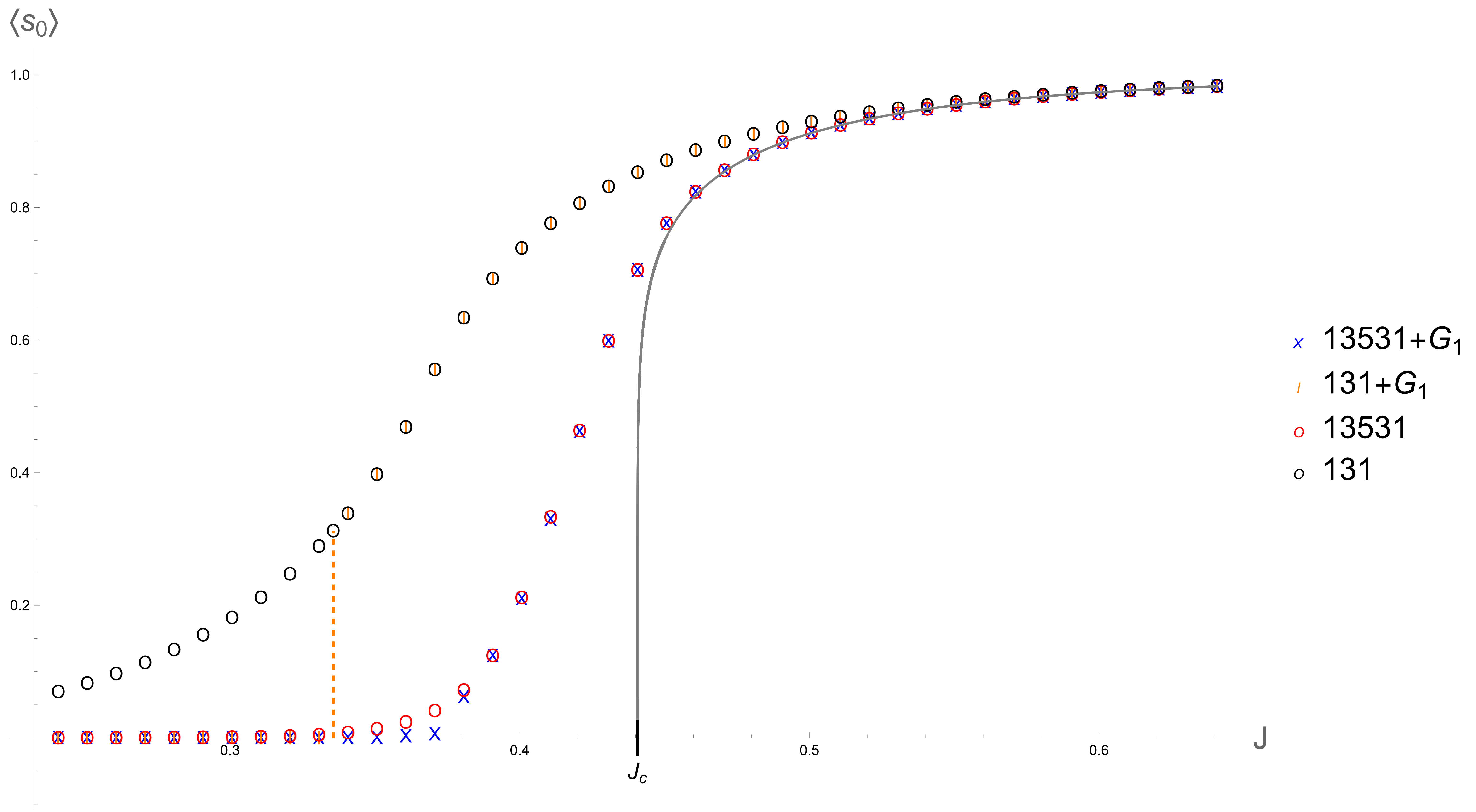
Results

2D Ising, h=0



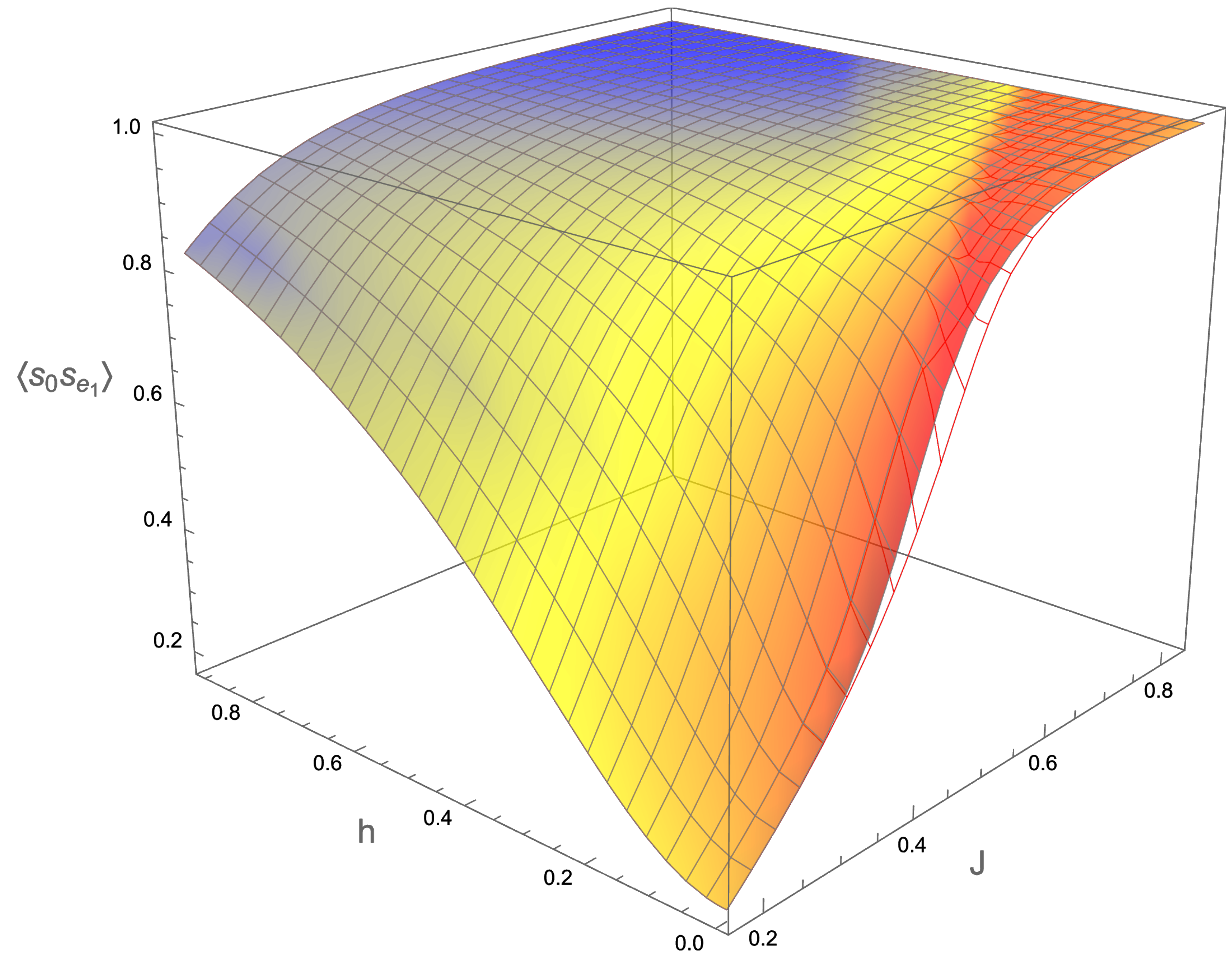
- Dramatic improvement by increasing size of diamond!

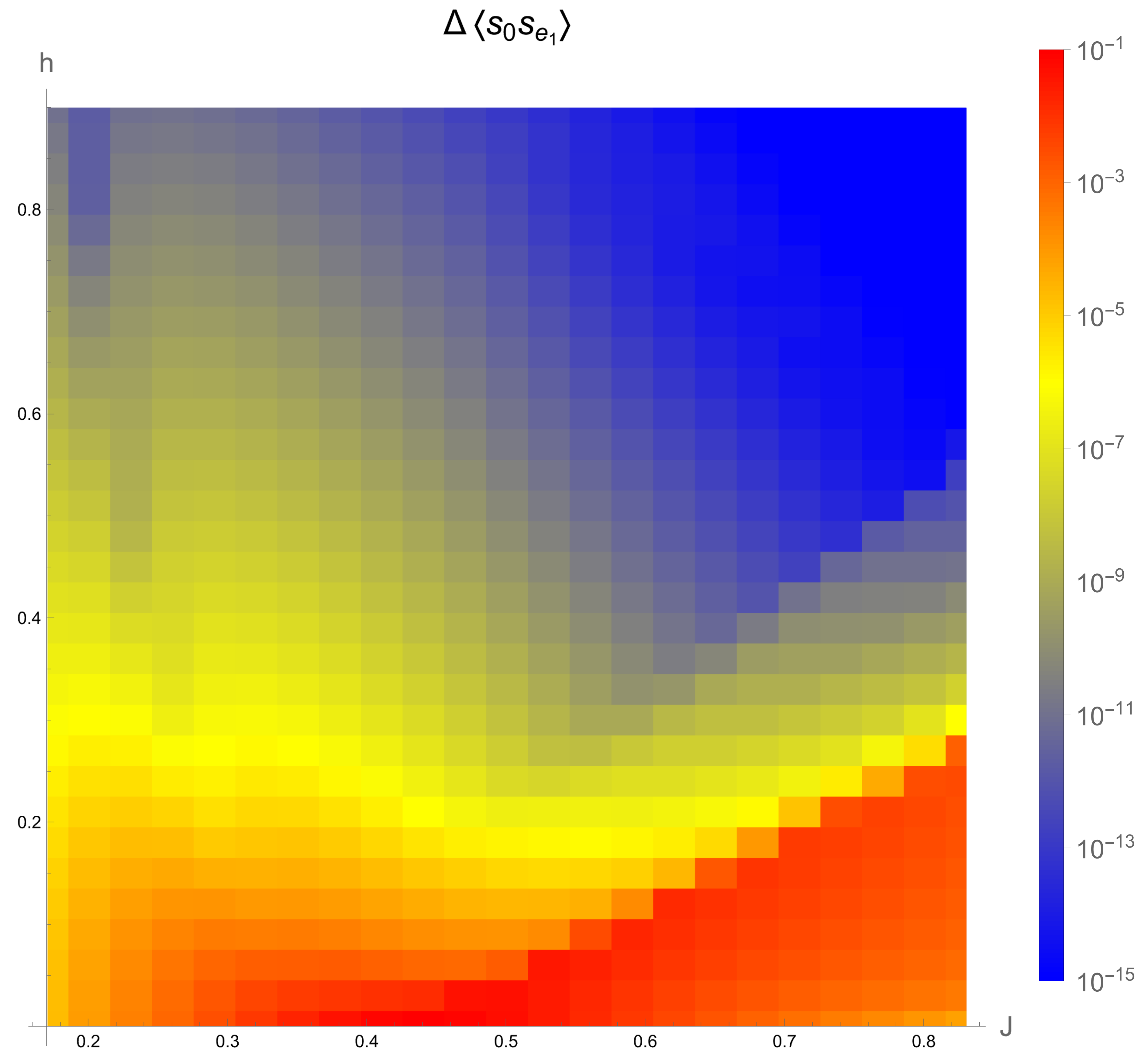
2D Ising, spontaneous magnetization



- Only upper bound
- 1st Griffiths inequality plays a role, but appears to be not essential

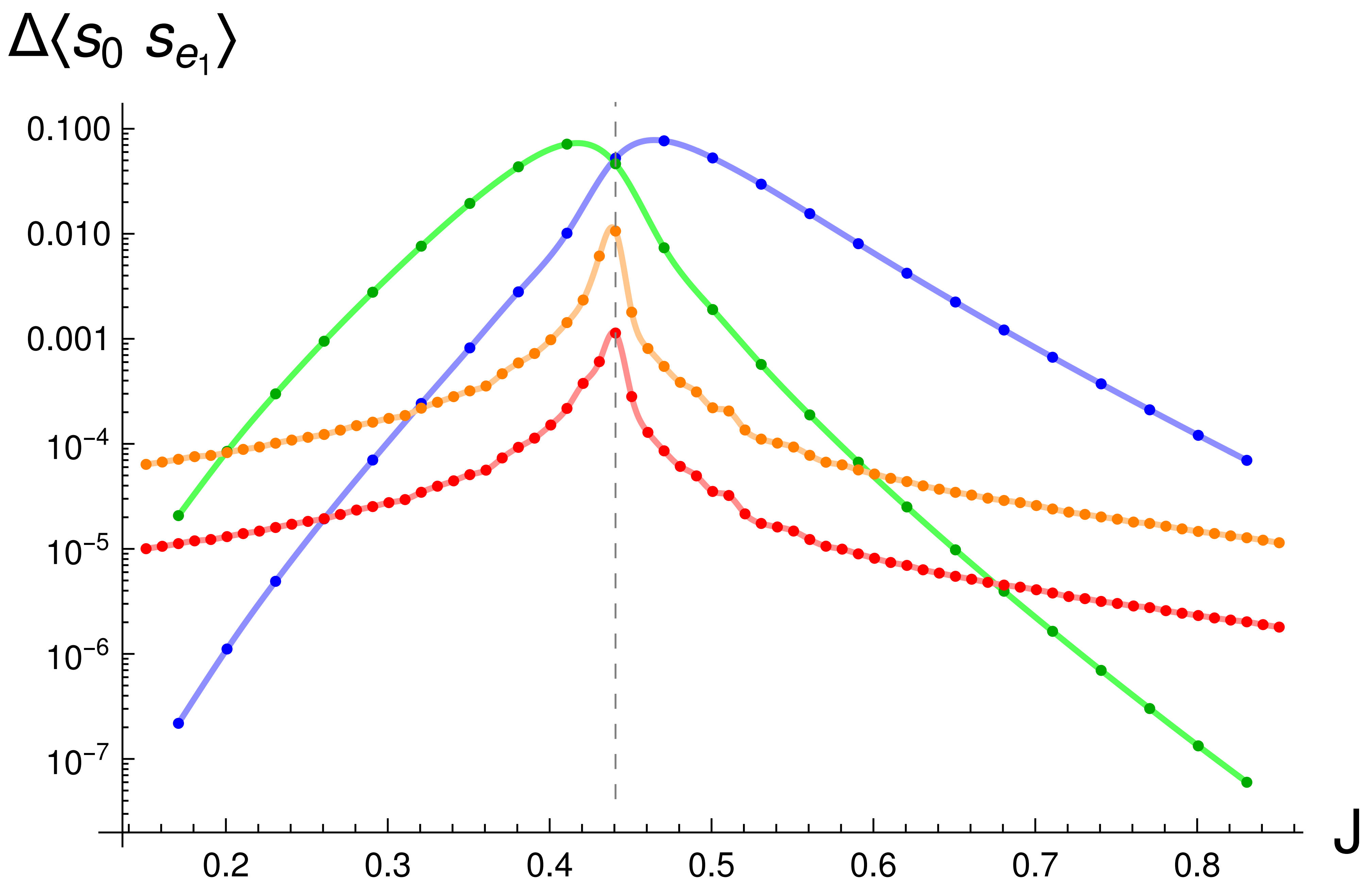
2D Ising, $h \neq 0$



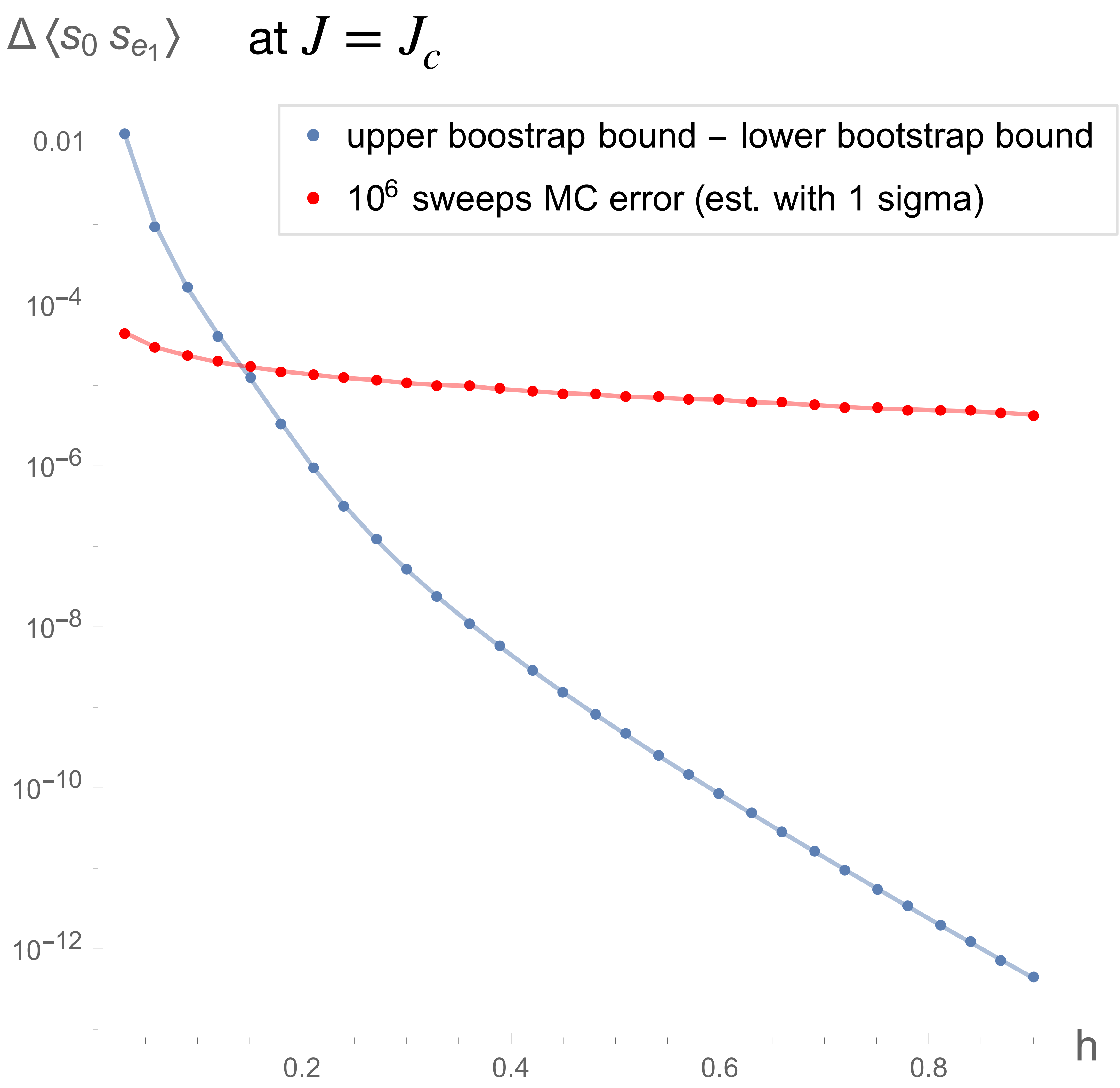


- 13531 diamond bootstrap

2D Ising

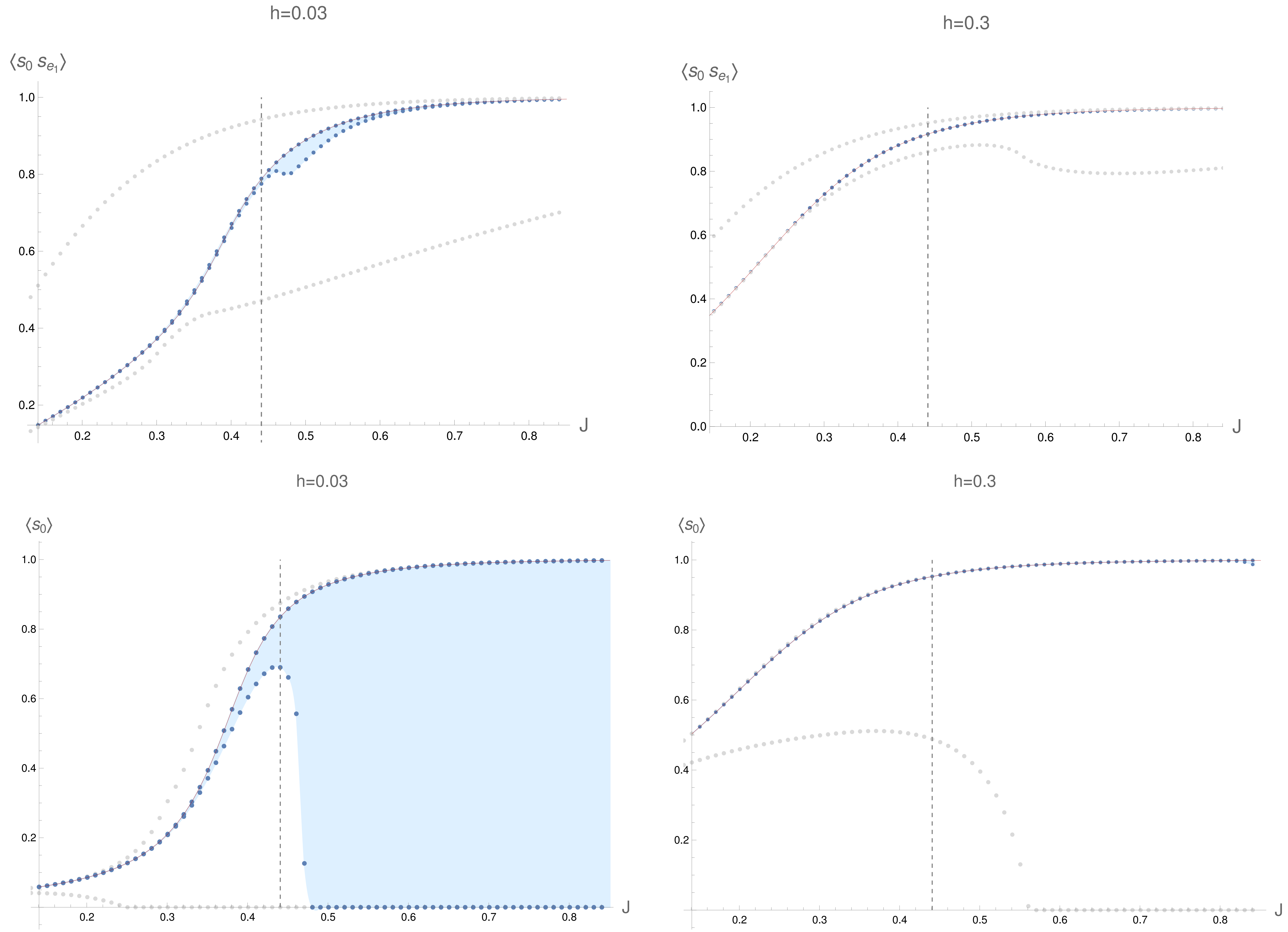


- exact – lower bound
- upper bound – exact
- 10^4 sweeps MC error (est. with 1 sigma)
- 10^6 sweeps MC error (est. with 1 sigma)



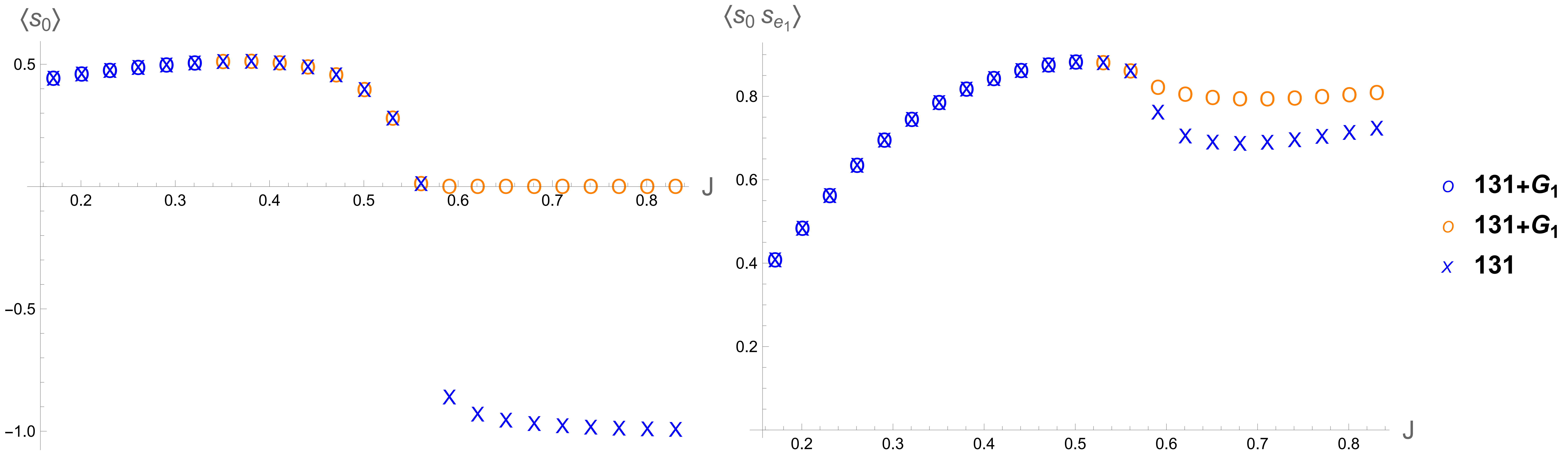
- 13531 diamond bootstrap
- MC on a 200×200 lattice, 10^6 Metropolis sweeps

2D Ising, h≠0 fixed



2D Ising, Griffiths 2nd inequality

$h = 0.3$



- Orange circles: 2nd Griffiths inequality G_2 is violated ~ when bound looks non-monotonic.

G_2 inequalities

$$\langle \underline{s}_A \underline{s}_B \rangle - \langle \underline{s}_A \rangle \langle \underline{s}_B \rangle \geq 0$$

- Some can be phrased as positive-semidefinite matrix:
  Namely, those with $B = A^g$ for some $g \in G$ of the symmetry group G, so that $\langle \underline{s}_A \underline{s}_{A^g} \rangle - \langle \underline{s}_A \rangle^2 \geq 0$, or

$$\begin{pmatrix} 1 & \langle \underline{s}_A \rangle \\ \langle \underline{s}_A \rangle & \langle \underline{s}_A \underline{s}_{A^g} \rangle \end{pmatrix} \geq 0$$

But do not appear to improve bounds.

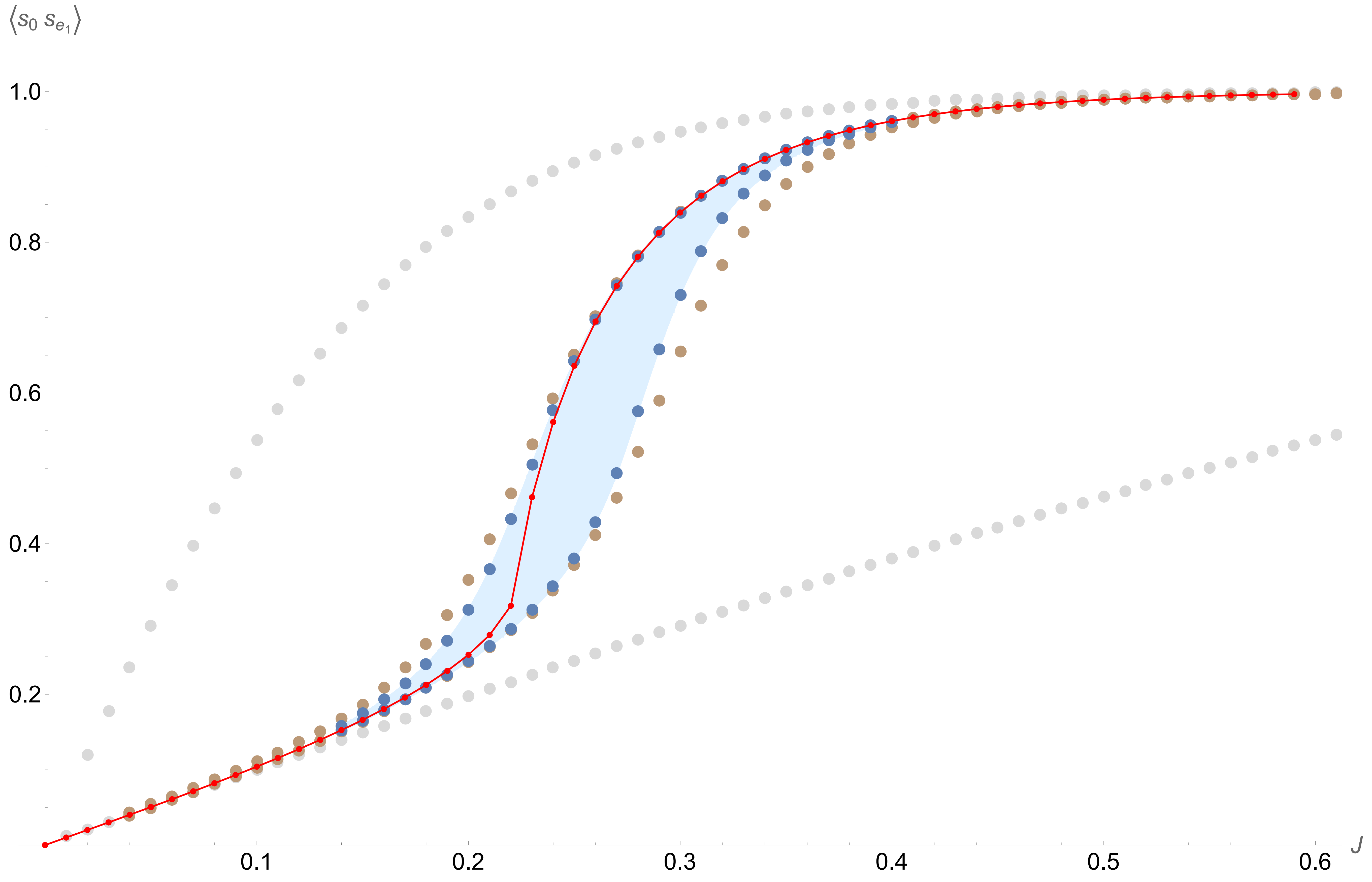
- Not others, for example in the 2D 131 diamond

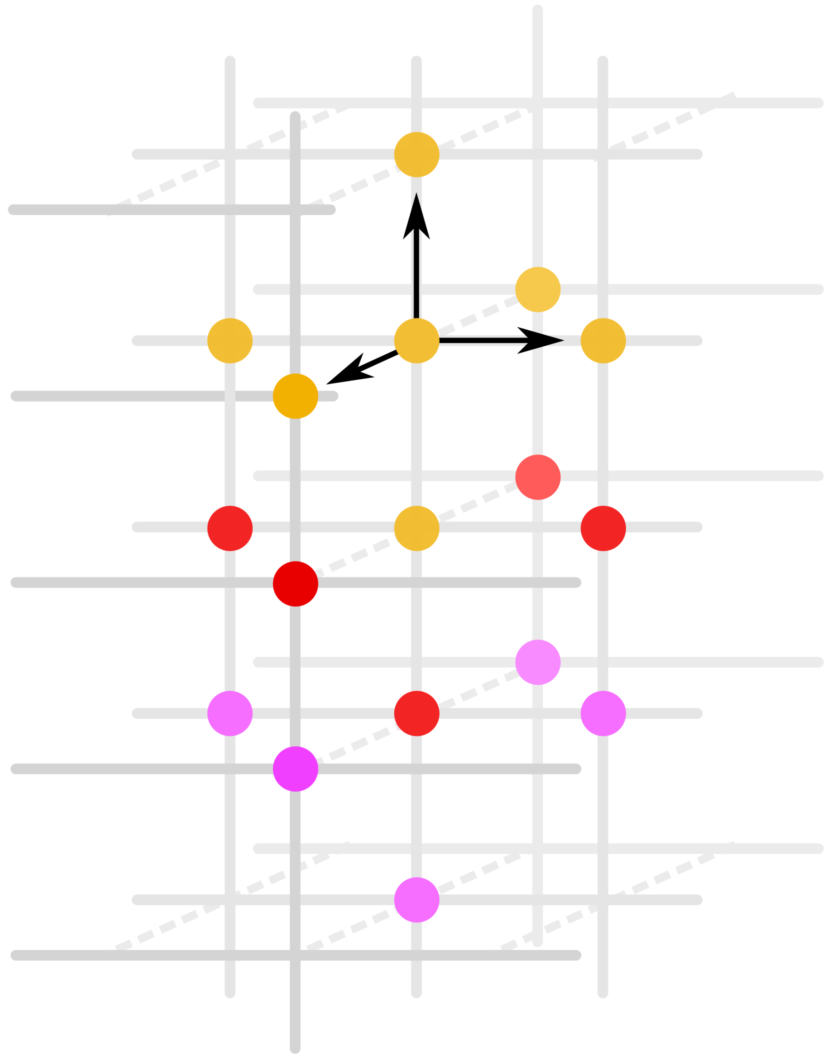
$$\langle s_{-e_2} s_0 s_{e_2} s_{e_1} \rangle - \langle s_0 s_{e_1} \rangle \langle s_{-e_2} s_{e_2} \rangle \geq 0, \quad \langle s_{e_2} \rangle - \langle s_{-e_1} s_{e_1} s_{e_2} \rangle \langle s_{-e_1} s_{e_1} \rangle \geq 0 \quad \text{etc.}$$

Some are violated! So we do expect to improve our bounds.

- A naive relaxation did not improve bounds (didn’t try too hard…)

3D Ising, h=0



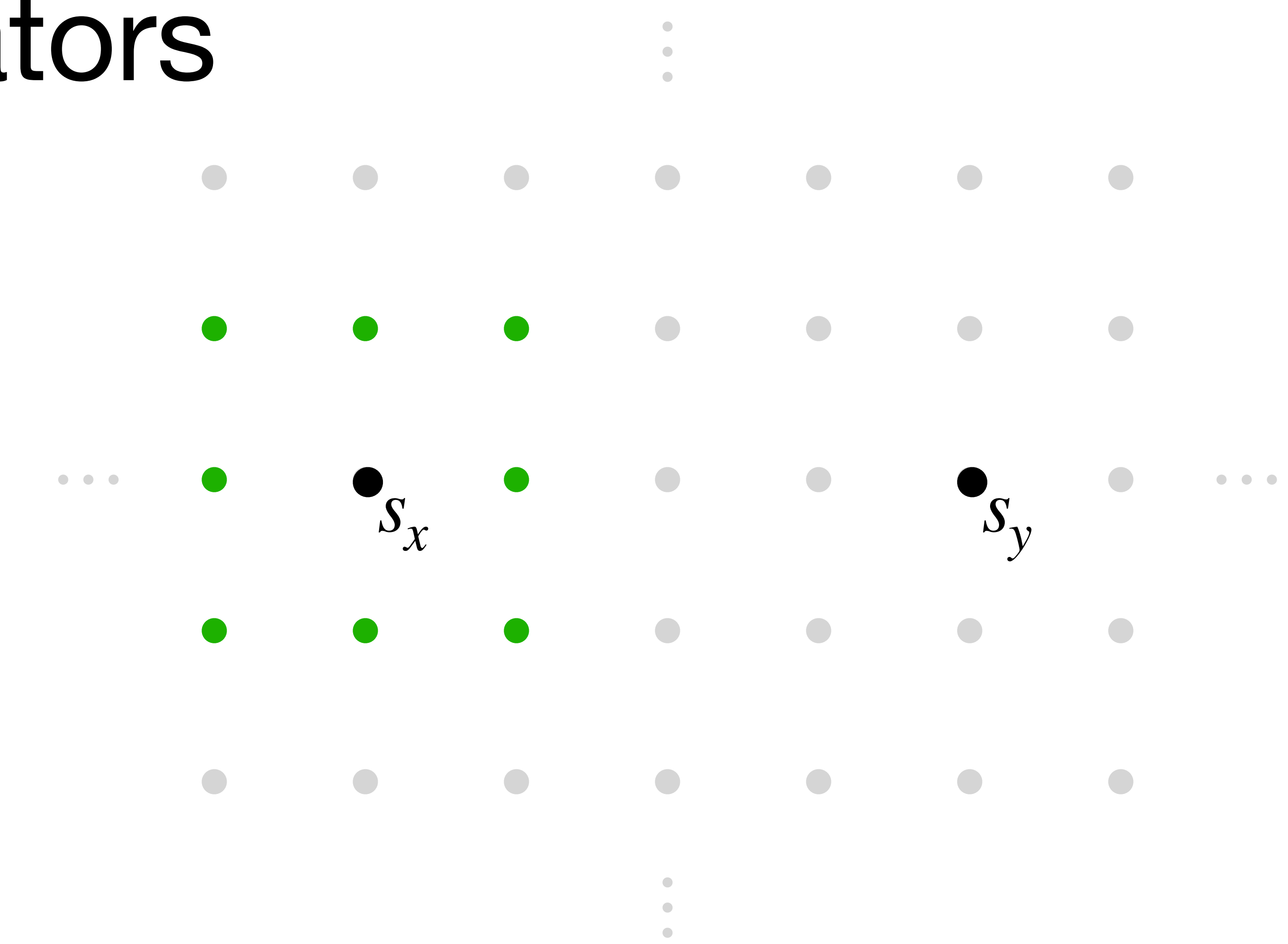


- 151 diamond
- 1551 diamond
- 15551 diamond, with reflection positivity matrices truncated to 100×100
- MC on 100^3 lattice

Future Directions

- Improve the algorithm
  - Subset of spin configurations that are more important
  - Null state relations
- More inequalities
  - Incorporate G_2 inequalities (non-convex)
  - Simon-Lieb inequalities - long-distance spin correlators

$$\langle s_x s_y \rangle \leq \sum_{z \in B} \langle s_x s_z \rangle \langle s_z s_y \rangle$$

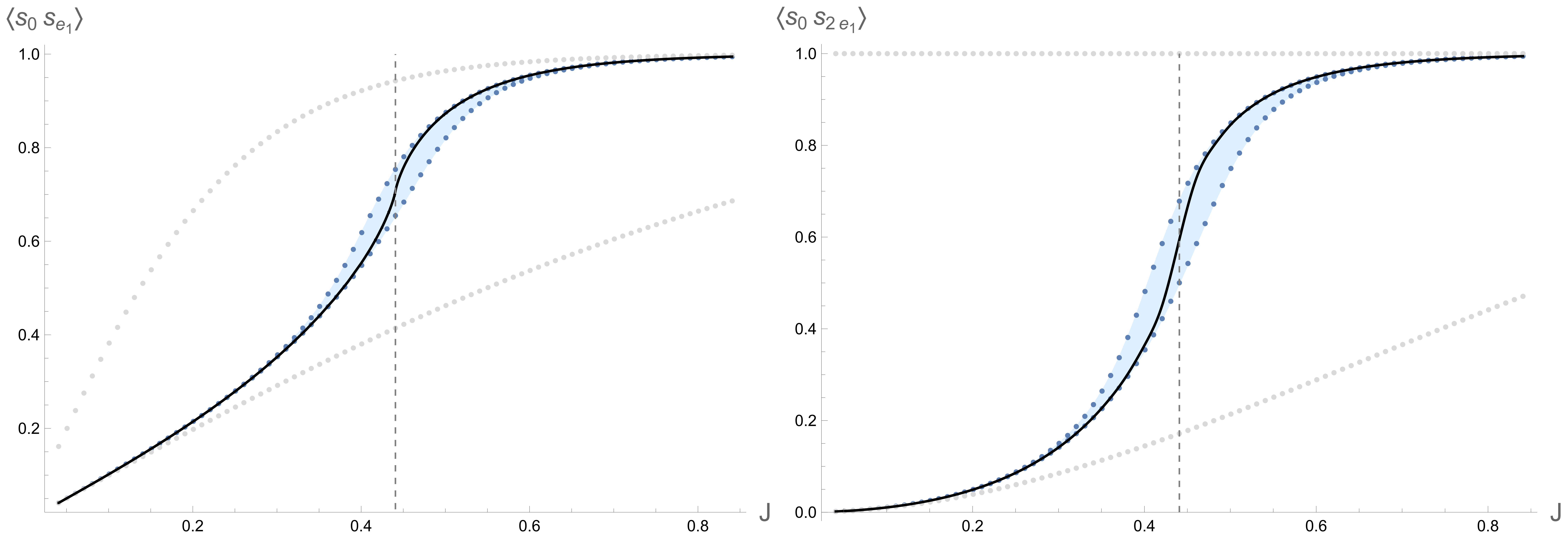


  - Aizenman-Lebowitz inequality
  - More!

Future Directions

- Theories with fermions
- Incorporate RG block-spin transformations (criticality)
- Systematic understanding of the convergence of bounds
- Gauge theories (see [Kruczenski talk] [Kazakov-Zheng] for pure YM)
- Study lattice defects
- Combine with the conformal bootstrap
- …

2D Ising, h=0



⟨s0 se1⟩
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
J
⟨s0 s2 e1⟩
1.0
0.8
0.6
0.4
0.2
0.0
0.2
0.4
0.6
0.8
J